MATH 223: Linear Algebra

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1 2018/01/9

2 2018/01/11

$v \subseteq \mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ if

1. $\vec{O} \in V$ ie V is non empty
2. $\vec{u} + \vec{v} \in V$ whenever $\vec{u} \in V + \vec{\alpha} \in V$
3. $\alpha\vec{u} \in V$ whenever $\vec{u} \in V, \vec{\alpha} \in \mathbb{R}$

A subspace V of $\mathbb{R}$ has a basis
ie a family $\{\overrightarrow{u_1}, \overrightarrow{u_2}, ..., \overrightarrow{u_k}\}$ of vectors in V such that $\{\overrightarrow{u_1}, \overrightarrow{u_2}, ..., \overrightarrow{u_k}\}$ is a spanning set of V
A spanning set of V is a set such that every vector in V is a linear combination of that set
ie whenever $\alpha_1\overrightarrow{u_1} + \alpha_2\overrightarrow{u_2} + ... + \alpha_k\overrightarrow{u_k} = 0$
if $A\alpha = 0$, rank of A is $k(\leq n)$, where $k =$ dimension of V
Examples

1. $E = \left\{ \overrightarrow{u} = \begin{bmatrix} t \\ 2t+s \\ 1 \end{bmatrix}, t \in \mathbb{R}, s \in \mathbb{R} \right\} \subseteq R^3$ * E is not a subspace of R^3 as the 0 matrix is not included
2. $F = \left\{ \overrightarrow{u} = \begin{bmatrix} t+s \\ 2t+s' \\ 1 \end{bmatrix}, t, s' \in \mathbb{R} \right\} \subseteq R^3$

   $\overrightarrow{u} \in F \Rightarrow \begin{bmatrix} t+s \\ 2t=s' \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + s'\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

   $F = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$ (linearly independent)
3. let $A = \begin{bmatrix} 1,1 \\ 2,-1 \\ 0,0 \end{bmatrix} \rightarrow \begin{bmatrix} 1,0 \\ 0,1 \\ 0,0 \end{bmatrix}$

   Rank$(A) = 2$: therefore $\left\{ \begin{bmatrix} 1,1 \\ 2,-1 \\ 0,0 \end{bmatrix} \begin{bmatrix} 1,0 \\ 0,1 \\ 0,0 \end{bmatrix} \right\}$ is linearly independent.

3 2018/01/16

3.1 Diagonalization

$T : R^2 \to R^2$

projection onto the line

$A = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

x + y = 0

A is diagonalizable, ie $A + P \cdot D \cdot P^{-1}$

where $D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

Let $\overrightarrow{u_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\overrightarrow{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

The canonical basis of $\mathbb{R}^2$ is $B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \overrightarrow{i}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \overrightarrow{j} \right\}$

Note that $B_1 = \{\overrightarrow{u_1}, \overrightarrow{v_1}\}$ is also a basis of $\mathbb{R}^2$

A is the standard matrix of T, it is in fact the matrix of T through the canonical basis of B

a vector $\overrightarrow{u} \in \overrightarrow{\mathbb{R}^2}$ has coordinates $\begin{bmatrix} x \\ y \end{bmatrix}$ with respect to B.

The coordinates of $T(\overrightarrow{u})$ with respect to B is $A\begin{bmatrix} x \\ y \end{bmatrix} = A\left(P\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right)$

Let $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ be the coordinates of $\overrightarrow{u}$ with respect to B_1

$\overrightarrow{us} = \overrightarrow{x_i} + \overrightarrow{y_j} = x_1\overrightarrow{u_1} + y_1\overrightarrow{v_1}$

$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = P\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$

$P^{-1}AP\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = D\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$

D is the matrix of the linear transformation T through the basis B_1

3.2 Vector Spaces

Let K be a field ($K = \mathbb{R}, K = \mathbb{C}$) Let V be a nonempty set V is equipped with 2 operations

Additions	if $\overrightarrow{u} \in \overrightarrow{v}, \overrightarrow{v} \in V$, then sum $\overrightarrow{u} + \overrightarrow{v}$ is defined
Scalar Multiplication	if $\overrightarrow{u} \in V, \alpha \in \mathbb{R}, \alpha\overrightarrow{u}$ is defined

V is called a vector space (over K) if the following properties hold:

- A_1 whenever $\overrightarrow{u}, \overrightarrow{v} \in V, \overrightarrow{u} + \overrightarrow{v} \in V$
- A_2 whenever $\overrightarrow{u}, \overrightarrow{v} \in V, \overrightarrow{u} + \overrightarrow{v} = \overrightarrow{v} + \overrightarrow{u}$
- A_3 whenever $\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w} \in V, (\overrightarrow{u} + \overrightarrow{v}) + \overrightarrow{w} = \overrightarrow{u} + (\overrightarrow{v} + \overrightarrow{w})$
- A_4 there exists a special vector in V called the zero vector, denoted by $\overrightarrow{0}$ such that whenever $\overrightarrow{u} \in V$, $\overrightarrow{u} + \overrightarrow{0} = \overrightarrow{0} + \overrightarrow{u} = \overrightarrow{u}$
- A_5 Given $\overrightarrow{u} \in V$, there exists $\overrightarrow{w} \in V$ such that $\overrightarrow{u} + \overrightarrow{w} = \overrightarrow{w} + \overrightarrow{u} = \overrightarrow{0}$
  $\overrightarrow{w}$ is denoted by $-\overrightarrow{u}$

- S_1 $\forall \alpha \in K, \forall \overrightarrow{u} \in V, \alpha vecu \in V$
- S_2 $1 \cdot \overrightarrow{u} = \overrightarrow{u}, 1 \in K(K = \mathbb{R}), \overrightarrow{u} \in V$
- S_3 whenever $\alpha, \beta \in K, \overrightarrow{u} \in V, \alpha\left(\beta\overrightarrow{u}\right) = (\alpha\beta)\overrightarrow{u}$
- S_4 whenever $\alpha, \beta \in K, \overrightarrow{u} \in V, (\alpha + \beta)\overrightarrow{u} = \alpha\overrightarrow{u} + \beta\overrightarrow{u}$
- S_5 whenever $\alpha \in K, \overrightarrow{u}, \overrightarrow{v} \in V$, $\alpha\left(\overrightarrow{u} + \overrightarrow{v}\right) = \alpha\overrightarrow{u} + \alpha\overrightarrow{v}$s

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Examples

1. $V = \mathbb{R}^n$ is a vector space over $K = \mathbb{R}$
2. let $M_{p\times q}$ be the set of all $p \times q$ matrices
   $M_{p\times q}$ is a vector space over $\mathbb{R}$
3. Let P be the set of all polynomials over $\mathbb{R}$
   $P_1, P_2 \in P$, $(P_1 + P_2)(x) = P_1(x) + P_2(x)\ \forall x \in \mathbb{R}$
   If $\alpha \in \mathbb{R} \in K, (\alpha P)(x) = \alpha P(x)\ \forall x \in \mathbb{R}$
4. Let 0 be the function such that $0(x) = 0\ \forall x$

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4 2018/01/18

4.1 Vector Spaces

Examples
Let D be a subset of $\mathbb{R}$ (D can be an interval for example)
Let $F(D)$ be the set of all real valued functions defined on D
For $f, g \in F(D)$, $\alpha, \beta \in \mathbb{R}$, $0 : D \to \mathbb{R}$

- $f + g : D \to \mathbb{R}$
- $(f + g)(x) = f(x) + g(x)$
- $(\alpha f)(x) = \alpha \cdot f(x)$
- $f + g = g + f$
- $(f + g) + h = f + (g + h)$
- $0(x) = 0$
- $f + 0 = f$
- $f + (-f) = 0$
- $1 \cdot f = f$
- $(\alpha + \beta)f(x) = \alpha f(x) + \beta f(x) = (\alpha f + \beta f)(x)$

Note that if we set $D = \mathbb{N}$
$F(\mathbb{N})$ = set of all real-valued sequences

4.2 Proposition

Let $(V, +, \cdot)$ be a vector space over K

1. The zero vector $\overrightarrow{0}$ in V is unique
2. Given $\overrightarrow{u} \in V$, the vector $\overrightarrow{-u}$ is unique
3. If $\alpha \overrightarrow{u} = 0$ then $\alpha = 0$ or $\overrightarrow{u} = \overrightarrow{0}$
4. $\overrightarrow{-u} = (-1)\overrightarrow{u}$

Proof

1. Let $\overrightarrow{0_1}$ and $\overrightarrow{0_2}$ be two vectors such that

$$\begin{cases} \overrightarrow{u} + \overrightarrow{0_1} = \overrightarrow{0_1} + \overrightarrow{u} = \overrightarrow{u} \quad \forall \overrightarrow{u} \\ \overrightarrow{u} + \overrightarrow{0_2} = \overrightarrow{0_2} + \overrightarrow{u} = \overrightarrow{u} \quad \forall \overrightarrow{u} \end{cases} \tag{1}$$

   It follows that
   $\overrightarrow{0_1} = \overrightarrow{0_1} + \overrightarrow{0_2} = \overrightarrow{0_2}$

2. Let $\overrightarrow{u} \in V$ and let $\overrightarrow{w_1}$ and $\overrightarrow{w_2}$ be two vectors such that
   $\overrightarrow{u} + \overrightarrow{w_1} = \overrightarrow{0}$
   $\overrightarrow{u} + \overrightarrow{w_2} = \overrightarrow{0}$

$$\begin{aligned} \overrightarrow{u} + \overrightarrow{w_1} &= \overrightarrow{0} \\ \overrightarrow{w_2} + (\overrightarrow{u} + \overrightarrow{w_1}) &= \overrightarrow{w_2} + \overrightarrow{0} \\ (\overrightarrow{w_2} + \overrightarrow{u}) + \overrightarrow{w_1} &= \overrightarrow{w_2} \qquad \text{associativity} \\ 0 + \overrightarrow{w_1} &= \overrightarrow{w_2} \\ \overrightarrow{w_1} &= \overrightarrow{w_2} \end{aligned} \tag{2}$$

3. Suppose $\alpha \overrightarrow{u} = 0$ If $\alpha \neq 0$

$$\begin{aligned} \frac{1}{\alpha} &\in K \quad K = \mathbb{R} \\ \frac{1}{\alpha}(\alpha \overrightarrow{u}) &= \frac{1}{\alpha}\overrightarrow{0} = \overrightarrow{0} \\ (\frac{1}{\alpha}\alpha)\overrightarrow{u} &= \overrightarrow{0} \quad \text{ie} 1 \cdot \overrightarrow{u} = \overrightarrow{u} = 0 \end{aligned} \tag{3}$$

4. $-\overrightarrow{u} = (-1)\overrightarrow{u}$

$$\begin{aligned} 1 + (-1) &= 0 \\ (1 + (-1))\overrightarrow{u} &= 0\overrightarrow{u} = \overrightarrow{u} \\ 1\overrightarrow{u} + (-1)\overrightarrow{u} &= \overrightarrow{0} \\ \overrightarrow{u} + (-1)\overrightarrow{u} &= \overrightarrow{0} \\ \therefore \qquad (-1)\overrightarrow{u} &= \overrightarrow{-u} \end{aligned} \tag{4}$$

4.3 Subspaces

Let $(V, +\cdot)$ be a vector space over K
Let E be a subset of $V (E \subseteq V)$
$(E, +\cdot)$ is called a subspace of $(V, +, \cdot)$
if $(E, +, \cdot)$ is a vector space over K.
<u>Proposition</u>
E is a subspace of V if the following properties hold:

1. $\overrightarrow{0} \in E$

2. Whenever $\overrightarrow{u}, \overrightarrow{v} \in E \qquad \overrightarrow{u} + \overrightarrow{v} \in E$
3. Whenever $\overrightarrow{u} \in E, \alpha \in K \qquad \alpha\overrightarrow{u} \in E$

Notice that $E \subseteq V$ is a subspace of V iff E is nonempty and $\alpha\overrightarrow{u} + \beta\overrightarrow{v} \in E$ whenever $\overrightarrow{u}, \overrightarrow{v} \in E, \alpha, \beta \in K$

Examples

1. Let $C([0,1])$ be the set of all continuous functions on $[0,1]$
   $C([0,1]) \subseteq F([0,1]) \leftarrow$ vector space
   The function $f : [0,1] \to \mathbb{R}$
   $f \in C([0,1])$(nonemptiness)
   If f and g are continuous on $[0,1]$, so if $f+g$, as well as $\alpha f \ \forall\alpha$
   $C[0,1]$ is a subspace of $F([0,1])$
2. Let $E = \left\{A \in M_{2\times 2} \mid A = A^T\right\}$
   Note that $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in E$
   whenever $A, B \in E$, $(A+B)^T = A^T + B^T = A + B$
   $A + B \in E$
   Also $(\alpha A)^T = \alpha A^T = \alpha A$
   ie $\alpha A \in E$
   E is a subspace of $M_{2\times 2}$

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5 2018/01/23

5.1 Subspaces

Examples

1. Let $E = \{p = P_3, \text{ such that } p(1) = 2\}$
   E is a nonempty subset of $P_3(p(x) = 2x \in E)$. But E is neither stable under addition nor stable under scalar multiplication.

   Ex $p_1(x) = 2x \in E$, but $(4p_1)(x) = 8x \notin E$.
   $\therefore E$ is not a subspace

2. Let $E = \{p \in P_3 \mid p(0) \geq 0\}$
   The zero polynomial $(0) \in E$
   let $p_1 \in E, p_2 \in E, (p_1 +_2)(0) = p_1(0) + p_2(0) \geq 0$
   $p1 + p_2 \in E$
   However, E is not stable under scalar multiplication.
   Ex $p(x) = x + 1 \in E \leftarrow p(0) = 1 \geq 0$
   if $\alpha < 0$, then $\alpha p(0) = \alpha < 0 \rightarrow \alpha p \notin E$
3. If A is a $n \times m$ matrix
   $\text{Null}(A) = \{x \in \mathbb{R}^m \mid AX = 0\}$
   $\text{Null}(A)$ is a subspace of $\mathbb{R}^m$
   Proof

   $X = 0 \in \text{Null}(A)$ since $A0 = 0$ Let $X_1, X_2 \in \text{Null}(A)$
   $A(X_1 + X_2) = AX_1 + AX_2 = 0 + 0 = 0$
   If $X \in \text{Null}(A), \alpha \in \mathbb{R}$
   $\alpha X \in \text{Null}(A)$ bc
   $A(\alpha X) = \alpha(AX) = \alpha 0 = 0$

Let $(V, +, \cdot)$ be a vector space on $\mathbb{R}$.
Let $\overrightarrow{u_1}, \overrightarrow{u_2}, ..., \overrightarrow{u_n}$ be n vector in V
Proposition
The subset $E \subseteq V$ of all linear combinations (lc) of $\overrightarrow{u_1}, \overrightarrow{u_2}, ..., \overrightarrow{u_n}$ is a subspace of V, and is denoted
$E = \text{span}\{\overrightarrow{u_1}, \overrightarrow{u_2}, ..., \overrightarrow{u_n}\}$
Proof

1. $\overrightarrow{0} \in E$ bc $\overrightarrow{0} = 0\overrightarrow{u_1} + 0\overrightarrow{u_2} + ... + 0\overrightarrow{u_n}$
2. E is stable under addition
   Let $\overrightarrow{v} = \sum_{i=1}^{n} \alpha_i \overrightarrow{u_i} \in E$
   $\overrightarrow{w} = \sum_{i=1}^{n} \beta_i \overrightarrow{u_1} \in E$
   $\overrightarrow{v} + \overrightarrow{w} = \sum_{i=1}^{n} (\alpha_i + \beta_i) \overrightarrow{u_i} \in E$
3. E is stable under scalar multiplication
   $\overrightarrow{u} = \sum_{i=1}^{n} \alpha_i \overrightarrow{u_i} \in E$ and $\beta \in \mathbb{R}$
   $\beta \overrightarrow{u} = \beta(\sum_{i=1}^{n} \alpha_i \overrightarrow{u_i}) = \sum_{i=1}^{n} (\beta \alpha_i) \overrightarrow{u_i} \in E$

Examples

1. Let A be a $n \times m$ matrix and $C_1, C_2, ..., C_m$ are the columns of A (each column $\in \mathbb{R}^n$). span $\{C_1, C_2, ..., C_m\}$ is a vector subspace of $\mathbb{R}^n$, called the column space of A and denoted Col$(A)n$.
   Similarly, the row space of A is Row$(A) = \text{Col}(A^T)$ is a subspace of $\mathbb{R}^m$.
2. $E = P_3$
   $p \in P_3), p(x) = ax^3 + bx^2 + cx + d$
   $P_3 = \text{span}\,\{x^3, x^2, x, 1\}$
3. $E = \{p \in P_3 \mid p(2) = 0\}$ is a subspace of P_3
   If $p \in E, p(x) = 0$
   ie $p(x) = (x-2)q(x)$ where $q(x) \in P_2$
   $p(x) = (x-2)(ax^2 + bx + c) = ax^2(x-2) + bx(x-2) + c(x-2) \quad a, b, c \in \mathbb{R}$
   $E = \text{span}\,\{x^2(x-2), x(x-2), x-2\}$
   $p \in P_3, p(x) = sum_{k=0}^3 \frac{f^{(k)}(2)}{k!}(x-2)^k$
   if $p \in E, p(2) = 0$

$$\begin{aligned} p(x) &= \sum_{k=0}^{3} \frac{p^{(k)}(2)}{k!}(x-2)^k \\ &= \frac{p^{(1)}(2)}{1!}(x-2) + \frac{p^{(2)}(2)}{2!}(x-2)^2 + \frac{p^{(3)}(2)}{3!}(x-2)^3 \end{aligned} \tag{5}$$

Proposition

Let E be a subspace of V
Let $F_1 = \{\overrightarrow{u_1}, \overrightarrow{u_2}, ..., \overrightarrow{u_n}\}$ be a subset of vectors in V
$F_2 = \{\overrightarrow{v_1}, \overrightarrow{v_2}, ..., \overrightarrow{v_n}\}$ be a subset of vectors in V
F_1 and F_2 are both spanning sets of the same subspace E of V iff every vector in F_1 is a lc of vectors in F_2 and every vector in F_2 is a lc of vectors in F_1.

6 2018/01/25

Wasn't there

7 2018/01/30

7.1 Linear Independence

7.1.1 Properties

- If a subset $\overrightarrow{u_1}, \overrightarrow{u_2}, ..., \overrightarrow{u_k}$ of vectors in V contains the zero vector $\left(\overrightarrow{u_i} = \overrightarrow{0} \text{ for some } i\right)$, then it is linearly dependent
- If $F = \{\overrightarrow{u_1}, \overrightarrow{u_2}, ..., \overrightarrow{u_k}\}$ is linearly independent, then any subset of F is linearly dependent

  \\TODO update
- if $F = \overrightarrow{u_1}, \overrightarrow{u_2}, ..., \overrightarrow{u_n}$ is linearly independent, and $\{\overrightarrow{u_1}, \overrightarrow{u_2}, ..., \overrightarrow{u_n}, \overrightarrow{u_{n+1}}\}$ is linearly independent, then $\overrightarrow{u_{n+1}} \in \text{span}\{\overrightarrow{u_1}, \overrightarrow{u_2}, ..., \overrightarrow{u_n}\}$

  Proof

  1. Without loss of generality, $\overrightarrow{u_1} = \overrightarrow{0}$
     Note that $2\overrightarrow{u_1} + 0\overrightarrow{u_2} + 0\overrightarrow{u_3} + ... + 0\overrightarrow{u_n} = \overrightarrow{0}$
     As there is a nonzero coefficient, there must be linear dependence.
  2. Let $F = \{\overrightarrow{u_1}, \overrightarrow{u_2}, ..., \overrightarrow{u_k}\}$ be linearly independent.
     Let F_1 be a subset of F containing k vectors, $k \leq n$
     $\sum_{i=1}^{n} \alpha_i \overrightarrow{u_i} = \overrightarrow{0} \Rightarrow \sum_{i=1}^{k} \alpha_i \overrightarrow{u_i} + 0\overrightarrow{u_{i+1}} + 0\overrightarrow{u_{i+2}} + ...0\overrightarrow{u_n} = \overrightarrow{0}$

  Since F is linearly independent, we must have $\alpha_1 = \alpha_2 = ... = \alpha_k = 0$
- Assume that $F = \{\overrightarrow{u_1}, \overrightarrow{u_2}, ..., \overrightarrow{u_k}\}$ is linearly independent and $\{\overrightarrow{u_1}, \overrightarrow{u_2}, ..., \overrightarrow{u_n}, \overrightarrow{u_{n+1}}\}$ is linearly dependent

  There exists a finite sequence $\alpha_1, \alpha_2, ..., \alpha_n, \alpha_{n+1}$, where not all values are zeroes, such that
  $\alpha_1\overrightarrow{u_1} + \alpha_2\overrightarrow{u_2} + ... + \alpha_n\overrightarrow{u_n} + \alpha_{n+1}\overrightarrow{u_{n+1}} = \overrightarrow{0}(*)$

  Claim $\alpha_{n+1} \neq 0$
  Assume $\alpha_{n+1} = 0$
  $\alpha_{n+1} = 0$ and $(*)$ yields
  $\alpha_1\overrightarrow{u_1} + \alpha_2\overrightarrow{u_2} + ... + \alpha_n\overrightarrow{u_n} = \overrightarrow{0}$
  which implies $\alpha_1 = \alpha_2 = ... = \alpha_n = 0$ (since F is linearly independent). That is a contraction, therefore $\alpha_{n+1} \neq 0$

(∗) can be rewritten as $\alpha_{n+1}\overrightarrow{u_{n+1}} = \alpha_1\overrightarrow{u_1} + \alpha_2\overrightarrow{u_2} + ... + \alpha_n\overrightarrow{u_n} = \sum_{i=1}^{n} \alpha_i\overrightarrow{u_i}$
$\overrightarrow{u_{n+1}} = \sum_{i=1}^{n} -\left(\frac{\alpha_i}{\alpha n+1}\right)\overrightarrow{u_i}$ ie $\overrightarrow{u_{n+1}} \in \text{span}\{\overrightarrow{u_1}, \overrightarrow{u_2}, ..., \overrightarrow{u_n}\}$

Proposition
If $F = \{\overrightarrow{u_1}, \overrightarrow{u_2}, ..., \overrightarrow{u_k}\}$ is linearly dependent, then one of the $\overrightarrow{u_i}$ can be written as the linear combination of the others.
Basis
Let V be a vector apces and E be a subspace of V. A basis of E is a family $F = \{\overrightarrow{u_1}, \overrightarrow{u_2}, ..., \overrightarrow{u_k}\}$ of vectors in E such that

1. $E = \text{span}\{\overrightarrow{u_1}, \overrightarrow{u_2}, ..., \overrightarrow{u_n}\}$
2. $F = \{\overrightarrow{u_1}, \overrightarrow{u_2}, ..., \overrightarrow{u_k}\}$ is linearly independent

Examples

1. $V = \mathbb{R}^3$ A basis of V is $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$
2. Let V be any vector space
   $E = \left\{\overrightarrow{0}\right\}$ does not have a basis because the only spanning set if $\left\{\overrightarrow{0}\right\}$ which is linearly dependent

   Lemma

   Let E be a subspace of V
   Let $F_1 = \{\overrightarrow{u_1}, \overrightarrow{u_2}, ..., \overrightarrow{u_k}\}$ be a spanning set of E
   Let $F_2 = \{\overrightarrow{v_1}, \overrightarrow{v_2}, ..., \overrightarrow{v_k}\}$ be a linearly independent subset of E
   then $m \geq n$

   Proof

   By contradiction
   Assume that $n > m$
   $[\overrightarrow{v_1}, \overrightarrow{v_2}, ..., \overrightarrow{v_n}] = [\overrightarrow{u_1}, \overrightarrow{u_2}, ..., \overrightarrow{u_n}]\, AX$
   $AX = 0$

   Fn $j = 1, 2, ..., n$
   $\overrightarrow{v_j} = \sum_{i=1}^{m} a_{ij}\overrightarrow{u_i}$ (because F_1 is a spanning set of E)
   Let $A = (a_{ij})_{1 \leq i \leq m 1 \leq j \leq n}$

$\sum_{j=1}^{m} x_j \overrightarrow{v_j} = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} \alpha_{ij} x_j \right) \overrightarrow{u_i} (*)$

A is $m \times n$ and $n > m$
Therefore, the homogeneous system $AX = 0$ has a non trivial solution.
Using the components of the nontrivial solution in $(*)$, we have $\sum_{j=1}^{n} x_j \overrightarrow{v_j} = \overrightarrow{0}$, but not all x_j are equal to 0.
ie F_2 is linearly dependent, which is a contradiction

Theorem
Let V be a vector space and E be a subspace of V such that $E \neq \left\{ \overrightarrow{0} \right\}$
All basis of E have the same number k of vectors; k is called the dimension of E
Notation $dim(E) = k$
Proof
Let $B_1 = \{setu1touk\}$ and $B_2 = \{setv1tovl\}$ be two basis of E. We have to prove that $l = k$

$$\left.\begin{array}{l} B_1 \text{ is a spanning set of } E \\ B_2 \text{ is linearly independent in } E \end{array}\right\} \Rightarrow k \geq l$$

$$\left.\begin{array}{l} B_2 \text{ is a spanning set of } E \\ B_1 \text{ is linearly independent in } E \end{array}\right\} \Rightarrow l \geq k$$

$k = l$
Remark
$E = \left\{ \overrightarrow{0} \right\}$ $dim(E) = 0$ $dim(\mathbb{R}^3) = 3$
Examples

1. P_n = set of all polynomials of order $\leq n$
   We have seen that $B = \{1, x, x^2, ..., x^n\}$ is a spanning set of P_n and is also linearly independent

   B is a basis of P_n
   therefore $dim(P_n) = n + 1$
2. $M_{2\times2}$ = set of all 2×2 matrices
   $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $E_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $E_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ $E_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$\{E_1, E_2, E_3, E_4\}$ is a basis of $M_{2\times2}$

$M = \begin{pmatrix} a & c \\ b & d \end{pmatrix} = aE_1 + bE_2 + dE_3 + cE_4$

$dim(M_{2\times2}) = 2 \times 2 = 4$

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8 2018/02/01

8.1 Basis & Dimensions

Examples

1. let $U = \{M \in M_{2\times2} \mid M = M^T\}$
   It is clear that U is a subspace of $M_{2\times2}$

   Basis of U

   Let $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ $\quad M^T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad M = M^T \Leftrightarrow b = c$

   $M \in U \Leftrightarrow M = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$

   $M = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

   Also $\{A_1, A_2, A_3\}$ is linearly independent.

   $\therefore \{A_1, A_2, A_3\}$ is a basis of U, ie $dim(U) = 3$

Lemma

(Fundamental)

If $\{\overrightarrow{u_1}, \overrightarrow{u_2}, ..., \overrightarrow{u_n}\}$ is a spanning set of U (a subspace of V) and $\{\overrightarrow{v_1}, \overrightarrow{v_2}, ..., \overrightarrow{v_k}\}$ is linearly independent in U then $k \leq n$.

Proposition

Let U be a subspace of V and $dim(U) = n$

1. Every spanning set of U has at least n elements

2. Every spanning set of U which contains n vectors is a basis of U

Proof

1. Let $\{\overrightarrow{u_1}, \overrightarrow{u_2}, ..., \overrightarrow{u_n}\} = B$ be a basis of U
   Let $F = \{\overrightarrow{v_1}, \overrightarrow{v_2}, ..., \overrightarrow{v_m}\}$ be a spanning set of U
   Note that B is linearly independent, by the fundamental lemma $m \geq n$
2. Let $F = \{\overrightarrow{v_1}, \overrightarrow{v_2}, ..., \overrightarrow{v_n}\}$ be a spanning set of U
   Claim F is also linearly dependent
   Suppose otherwise; one of the $\overrightarrow{v_i}$ is a lc of the other ones.
   WLOG $\overrightarrow{v_n}$ is a lc of $\overrightarrow{v_1}, \overrightarrow{v_2}, ..., \overrightarrow{v_{n-1}}$
   $U = \text{span}\{\overrightarrow{v_1}, \overrightarrow{v_2}, ..., \overrightarrow{v_n}\} = \text{span}\{\overrightarrow{v_1}, \overrightarrow{v_2}, ..., \overrightarrow{v_{n-1}}\}$
   Thus, a contradiction as every spanning set must have at least n elements. Therefore, F is linearly independent, and a basis of U.

Examples

1. T or F: $V = \text{span}\{x^2, x+1\}$
   False, $dim(P_2) = 3$, and every spanning set must have at least 3 elements.
2. T or F: $V = \text{span}\left\{\underbrace{x^2}_{P_1}, \underbrace{x+1}_{P_2}, \underbrace{x^2-x-1}_{P_3}, \underbrace{2x+3}_{P_4}\right\}$
   Note that $x^2 - x - 1$ is a lc of x^2 and $x + 1$
   Let $p(x) = ax^2 + bx + c \in P_2$
   Can we find $x_1, x_2, x_3 \in \mathbb{R}$ st. $p = x_1p_1 + x_2p_2 + x_3p_4$ (*)

$$\text{(*) implies}\begin{cases} x_2 + 3x_3 &= c \\ x_2 + 2x_3 &= b \\ x_1 &= a \end{cases} \tag{6}$$

$AX = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix}$

A is invertible, thus

$X = A^{-1}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ ie $\{P_1, P_2, P_4\}$ is a spanning set of P_2

Proposition

Let U be a subspace of V and $dim(U) = n$

1. Every linearly independent subset of U has at most n vectors
2. Any linearly independent subset of U which contains n elements is a basis of U

Proof

1. Use the fundamental lemma
2. Let $B = \{\overrightarrow{u_1}, \overrightarrow{u_2}, ..., \overrightarrow{u_n}\}$ is a basis of U and $F = \{\overrightarrow{v_1}, \overrightarrow{v_2}, ..., \overrightarrow{v_n}\}$ a linearly independent set in U
3. Claim F is also a spanning set o U
   Proof by contradiction
   $\overrightarrow{w}$ is not a lc of $\overrightarrow{v_1}, \overrightarrow{v_2}, ..., \overrightarrow{v_n}$ then $\{\overrightarrow{v_1}, \overrightarrow{v_2}, ..., \overrightarrow{v_n}, \overrightarrow{w}\}$ is linearly independent
   $\alpha_1\overrightarrow{v_1} + \alpha_2\overrightarrow{v_2} + ... + \alpha_k\overrightarrow{v_k} = \overrightarrow{0}$
   This is a linearly independent subset with $n + 1$ elements, which is a contradiction

Proposition

Let U and W be two subspaces of a vector space V

1. If $U \subseteq W$ then $dim(U) \leq dim(W)$
2. If $U \subseteq W$ and $dim(U) = dim(W)$ then $U = W$ Proof
   (a) A basis of U is a linearly independent set of vectors in W, thus has at most $dim(W)$ vectors
   (b) $U \subseteq W \qquad dim(U) = dim(W) = n$
   Let $B = \{\overrightarrow{u_1}, \overrightarrow{u_2}, ..., \overrightarrow{u_n}\}$ be a basis of U; B is a linearly independent set of vectors in W and B has $n = dim(W)$ vectors. By the previous proposition, B is a basis of W.
   $\therefore W = \text{span}\,\{\overrightarrow{u_1}, \overrightarrow{u_2}, ..., \overrightarrow{u_n}\} = U$

Examples

1. $U = \{f \in F(N) \mid f(n + 2) = 3f(n + 1) - 2f(n)\}$
   $f_1(n) = 1 \quad (1, 1, 1, ...)$
   $f_2(n) = 2^n \quad (2^0, 2^1, 2^2, ...)$

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Example

$U = \{f \in F(\mathbb{N}) \mid f(n+2) - 3f(n+1) + 2f(n) = 0\}$

Note that U is the set of all sequences $\{x_n\}_{n\geq 0}$ such that $x_{n+2} - 3x_{n+1} + 2x_n = 0$

Note that if $f(n) = r^n \in U$, then $r = 1$ or $r = 2$

$f_1(n) = 1 \quad \forall n$ is an element of U

$f_2(n) = 2^n \quad \forall n$

$\{f_1, f_2\}$ is a basis of U

If $f \in U$ and $f(0) = f(1) = 0$, using the relation $f(n+2) - 3f(n+1) + 2f(n) = 0$, we can deduce that $f(n) = 0 \quad \forall n$.

$\{f_1, f_2\}$ is linearly independent

Suppose that $\alpha f_1 + \beta f_2 = 0$

\\TODO

$\{f_1, f_2\}$ is a spanning set of U

Let $f \in U$

$\exists a, b \in \mathbb{R}$ such that

$$\begin{cases} af_1(0) + bf_2(0) = f(0) \\ af_1(1) + bf_2(1) = f(1) \end{cases} \tag{7}$$

$\Leftrightarrow$

$$\begin{cases} a + b = f(0) \\ a + 2b = f(1) \end{cases} \tag{8}$$

$b = f(1) - f(0)$

$a = 2f(0) - f(1)$

Let $g(n) = f(n) - (2f(0) - f(1))f_1(n) - (f(1) - f(0))f_2(n)$

$g \in U$ and $g(0) = 0$, $g(1) = 0$

$\therefore$ using (*) $g(n) = 0 \quad \forall n$

ie $f(n) = (2f(0) - f(1))f_1(n) + (f(0) - f(1))f_2(n)$

Any sequence $\{x_n\}_n$ such that $x_{n+2} = 3x_{n+1} + 2x_n = 0$ can be written as

$$x_n = (2x_0 - x_1) + (x_0 - x_1)2^n$$

Exercises
Find a basis for each of the following subspaces

1. $U = \{f \in F(\mathbb{N}) \mid f(n+2) - 4f(n+1) + 4f(n) = 0\}$
2. $U = \{f \in F(\mathbb{N}) \mid f(n+2) - 5f(n+1) + 6f(n) = 0\}$

9.1 Direct Sum

Let V be a vector space and E, F are 2 subspaces of V
$E + F = \{\overrightarrow{u} = \overrightarrow{u_1} + \overrightarrow{u_2}, \overrightarrow{u_1} \in E, \overrightarrow{u_2} \in F\} \subseteq V$
Examples

1. $V = \mathbb{R}^2$
   $E = \text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix} = i\right\} \qquad F = \text{span}\left\{\begin{bmatrix}0\\1\end{bmatrix} = j\right\}$
   $E + F = \mathbb{R}^2$
2. $V = \mathbb{R}^3$
   $E = \text{span}\left\{i = \begin{bmatrix}1\\0\\0\end{bmatrix}, j = \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$ xy-plane
   $F = \text{span}\left\{j = \begin{bmatrix}0\\1\\0\end{bmatrix}, k = \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ yz-plane

Definition
Let V be a vector space and E and F be 2 subspaces of V
V is said to be the direct sum of E and F
(Notation: $= V = E \oplus F$)
If $V = E + F$ and $E \cap F = \left\{\overrightarrow{0}\right\}$

Proposition
If $V = E_1 \oplus E_2$, then every vector $\overrightarrow{u} \in V$ can be written uniquely as $\overrightarrow{u} = \overrightarrow{u_1} + \overrightarrow{u_2}$, where $\overrightarrow{u_1} \in E_1, \overrightarrow{u_2}, \in E_2$

Proof

$$\begin{aligned} \vec{u} &= \vec{u_1} + \vec{u_2} & \vec{u_1}, \vec{v_1} \in E_1 \\ &= \vec{v_1} + \vec{v_2} & \vec{u_2} + \vec{v_2} = \in E_2 \\ \vec{u_1} + \vec{u_2} & & = \vec{v_1} + \vec{v_2} \\ \vec{w} &= \underbrace{\vec{u_1} - \vec{v_1}}_{\in E_1} & = \underbrace{\vec{v_2} - \vec{u_2}}_{\in E_2} = \vec{0} \end{aligned} \tag{9}$$

$\vec{w} \in E_1, \vec{w} \in E_2, \vec{w} \in E_1 \cap E_2$
ie $\vec{w} = \vec{0}$

Theorem

Let V be a finite dimensional vector space
Assume that $V = E_1 \oplus E_2$
then $dim(V) = dim(E_1) + dim(E_2)$
More precisely, if $B_1 = \{\vec{u_1}, \vec{u_2}, ..., \vec{u_n}\}$ is a basis of E_1 and $B_2 = \{\vec{v_1}, \vec{v_2}, ..., \vec{v_m}\}$ is a basis of E_2, then $B = B_1 \cup B_2$ is a basis of V

Proof

$\because V = E_1 \oplus E_2, V = E_1 + E_2$
$\therefore B$ is a spanning set of V

$$\begin{aligned} \vec{u} &\in V \\ \vec{u} &= \underbrace{\vec{w_1}}_{\in E_1} + \underbrace{\vec{w_2}}_{\in E_2} \\ &= \sum_{i=1}^{n} \alpha_1 \vec{u_i} + \sum_{i=1}^{m} \beta_i + \vec{v_i} \end{aligned} \tag{10}$$

B is linearly independent $\left(\text{bc } E_1 \cap E_2 = \left\{\vec{0}\right\}\right)$

$$\begin{aligned} \sum_{i=1}^{n} \alpha_i \vec{u_i} + \sum_{i=1}^{m} \beta_i \vec{v_i} &= \vec{0} \Leftrightarrow \\ \sum_{i=1}^{n} \alpha_i \vec{u_i} = -\sum_{i=1}^{m} \beta_i \vec{v_i} &= \vec{w} \\ \vec{w} \in E_1 \cap E_2 &= \left\{\vec{0}\right\} \\ \vec{w} &= \vec{0} \end{aligned} \tag{11}$$

$$\sum_{i=1}^{n} \alpha_i \overrightarrow{u_i} = \overrightarrow{0} \Rightarrow \alpha_i = 0 \quad \forall i = 1, 2, ..., n \quad B_1 \text{ is a basis}$$

$$\sum_{i=1}^{m} \beta_i \overrightarrow{v_i} = \overrightarrow{0} \Rightarrow \beta_i = 0 \quad \forall i = 1, 2, ..., m \quad B_2 \text{ is a basis}$$

Examples

1. $E = \text{span}\{2 - x, 1 + x^2\}$ Find F such that
   $E \oplus F = P_2$
   $\because \{2 - x, 1 + x^2\}$ is linearly independent
   $\therefore dim(E) = 2$
   if $P_2 = E \oplus F$
   $3 = dim(P_2) = dim(E) + dim(F)$
   $dim(F) = 1$

   Let $p(x) = 1 \quad \forall x$
   $p \in P_2$, but $p \notin E$
   $F = \text{span}\{p\}$
   $P_2 = F \oplus E$

2. Let $V = M_{2\times 2}$
   $E = \left\{M \in M_{2\times 2} \mid M = M^T\right\}$
   Find F such that $E \oplus F = M_{2\times 2}$

$$\begin{aligned} M_1 &= A + A^T \\ M_1^T &= A^T + \left(A^T\right)^T = A^T + A = M_1 \\ M_2 &= A - A^T \\ M_2^T &= A^T - A = -M_2 \\ E &= \left\{M \in M_{n\times n} \mid M = M^T\right\} \\ F &= \left\{M \in M_{n\times n} \mid M^T = -M\right\} \\ M \in E \cap F &\Rightarrow M = 0 \end{aligned} \tag{12}$$

Let $A \in M_{n\times n}$

$$A = \underbrace{\frac{1}{2}(A + A^T)}_{\in E} + \underbrace{\frac{1}{2}(A - A^T)}_{\in F}$$

$M_{n\times n} = E \oplus F$

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$$S = \left\{M \in M_{n\times n} \mid M^T = M\right\} A = \left\{M \in M_{n\times n} \mid M^T = -M\right\} \begin{cases} M_{n\times n} = S \oplus A \\ dim(A) = \frac{n^2-n}{2} \\ dim(S) = \frac{n^2+n}{2} \end{cases} \quad (13)$$

$dim(E \oplus F) = dim(E) + dim(F)$
If $dim(E + F) = dim(E) + dim(F)$ then it is a direct sum.
Exercise
$dim(E + F) = dim(E) + dim(F) - dim(E \cap F)$
$(E \cap F)$ is a subspace of V whenever E and F are subspaces of V.

10.1 Coordinates

Let V be a vector space such that $dimm(V) = n$
Let $B = \{\overrightarrow{u_1}, \overrightarrow{u_2}, ..., \overrightarrow{u_n}\}$ be a basis of V.
Given $\overrightarrow{w}$ in V, $\overrightarrow{w}$ can be written uniquely as a linear combination of vectors in B.
ie $\overrightarrow{w} = \sum_{i=1}^{n} x_i \overrightarrow{u_i}$

Therefore the column-matrix $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ uniquely identifies $\overrightarrow{w}$.

$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is called the coordinate-vector of $\overrightarrow{w}$ relative to the basis B.

Examples

1. $M_{2\times 2} \qquad B = \left\{E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, E_4 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right\}$

   $M = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$

$U = \{A \in M_{\times 2} \mid A^T = A\}$

A basis of U is given by $B_1 = \left\{A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$

Coordinate vector of M relative to B

$M = E_1 + 3E_2 + 2E_3 + 3E_4 \leftrightarrow \begin{bmatrix} 1 \\ 3 \\ 2 \\ 3 \end{bmatrix}$

Coordinate vector of M relative to B_1

$M = A_1 + 3A_2 + 2A_3 \leftrightarrow \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ (Note that the order for which you write the basis is important)

2. Find a basis B of $U = \text{span}\left\{\underbrace{1+x}_{P_1}, \underbrace{3+x^2}_{P_2}, \underbrace{(x-1)^2}_{P_3}\right\}$ and find the coordinate vector of $p(x) = (x-1)^2$ relative to B.

$\{P_1, P_2, P_3\}$ is a spanning of U.

Linear Independence

$\alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 = 0$

$\underbrace{A}_{3\times 3} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \underbrace{0}_{3\times 1}$ where $A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$P_3 = -2P_1 + P_2$

$\{P_1, P_2\}$ is linearly independent

$B = \{P_1, P_2\}$ is a basis of U and coordinates of $P = P_3$ relative to B is $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

<u>Remark</u>

If V is a n-dimensional vector space over $\mathbb{R}$

$B = \{\overrightarrow{u_1}, \overrightarrow{u_2}, ..., \overrightarrow{u_n}\}$ is a basis of V

The map $T : V \to \mathbb{R}^n, T(U) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

where $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is the coordinate-vector of $\overrightarrow{u}$ relative to B

is an isomorphism $\overrightarrow{u} \leftrightarrow \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

10.2 Linear Transformations (Mapping)

Definition

Let V and W be two vector spaces over $\mathbb{R}$

A map (of function) $T : V \to W$ is called a linear transformation of the following properties hold.

1. $T(\overrightarrow{u_1} + \overrightarrow{u_2}) = T(\overrightarrow{u_1}) + T(\overrightarrow{u_2})$ whenever $\overrightarrow{u_1}, \overrightarrow{u_2} \in V$
2. $T(\alpha \overrightarrow{u}) = \alpha T(\overrightarrow{u})$ whenever $\overrightarrow{u} \in V \quad \alpha \in \mathbb{R}$

Remark

$T : V \to W$ is a linear transformation iff $T(\alpha_1 \overrightarrow{u_1} + \alpha_2 \overrightarrow{u_2}) = \alpha_1 T(\overrightarrow{u_1}) + \alpha_2 T(\overrightarrow{u_2})$ whenever $\overrightarrow{u_1}, \overrightarrow{u_2} \in V \quad \alpha_1, \alpha_2 \in \mathbb{R}$

or equivalently: $T(\sum_{i=1}^{n} \alpha_i \overrightarrow{u_i}) = \sum_{i=1}^{n} \alpha_i T(\overrightarrow{u_i})$

whenever $\alpha_i \quad i = 1, ..., n \in \mathbb{R} \qquad \overrightarrow{u_i} \quad i = 1, ..., n \in V$

Examples

1.

$$V = P$$
$$T : V \to R$$
$$T(p) = [p(0)]^2$$

$$\begin{aligned} p_1(x) &= x - 1 \\ p_2(x) &= x + 1 \\ T(p_1) &= (p_1(0))^2 = 1 \\ T(p_2) &= 1 \\ T(p_1 + p_2) &= 0 \end{aligned} \tag{14}$$

T is not a linear transformation

2. The coordinate-map $dim(V) = n$ and $B = \{\overrightarrow{u_1}, \overrightarrow{u_2}, ..., \overrightarrow{u_n}\}$ is a basis of V
   The map $T : V \to \mathbb{R}^n$
   $T(\overrightarrow{u}) = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} \leftarrow$ coordinates of $\overrightarrow{u}$ relative to B
   T is a linear transformation

3. $T : \mathbb{R}^p \to \mathbb{R}^q$
   $\underbrace{T(X)}_{q\times 1} = \underbrace{A}_{q\times p} \underbrace{X}_{p\times 1}$, where A is a $q \times p$ matrix, is a linear transformation
4. $T : P_2 \to P_2$
   $T(p)(x) = xp'(x) + \int_0^1 p(x)dx$
   $T(p_1+p_2)(x) = x(p_1'(x)+p_2'(x))+\int_0^1(p_1(x)+p_2(x))dx = xp_1'(x)+\int_0^1 p_1(x)dx+xp_2'(x)+\int_0^1 p_2(x)dx = T(p_1)(x) + T(p_2)(x)$
   $T(\alpha p)(x) = \alpha xp'(x) + \alpha\int_0^1 p(x)dx = \alpha T(p)(x)$

Proposition
Let $T : V \to W$ be a linear transformation

1. $T(\overrightarrow{O_V}) = \overrightarrow{O_W}$
2. Let E be a subspace of V
   $T(E) = \{T(\overrightarrow{u}),\ \text{where } \overrightarrow{u} \in E\}$ is a subspace of W
3. Let F be a subspace of W
   $T^{-1}(F) = \{\overrightarrow{u} \in V \mid T(\overrightarrow{u}) \in F\}$ is a subspace of V

   Proof

   (a)

$$\begin{gathered} \overrightarrow{u} \in V \\ \overrightarrow{u} + \overrightarrow{O_V} = \overrightarrow{u} \\ T(\overrightarrow{u} + \overrightarrow{O_V}) = T(\overrightarrow{u}) \\ T(\overrightarrow{u}) + T(\overrightarrow{O_V}) = T(\overrightarrow{u}) \end{gathered} \tag{15}$$

$\therefore T(\overrightarrow{O_V}) = \overrightarrow{O_W}$

(b) Let $E \subseteq V$ be a subspace of V

$T(E) = \{T(\overrightarrow{u}),\ \text{where } \overrightarrow{u} \in E\}$ (reverse in) $\overrightarrow{O_W}$

$\overrightarrow{O_V} \in E,\ \therefore \overrightarrow{O_W} = T(\overrightarrow{O_V}) \in T(E)$

Let $\overrightarrow{w_1}, \overrightarrow{w_2} \in T(E);\ \alpha_1, \alpha_2 \in \mathbb{R}$

$\overrightarrow{w_1} = T(\overrightarrow{u_1})$ where $\overrightarrow{u_1} \in E$

$\overrightarrow{w_2} = T(\overrightarrow{u_2})$ where $\overrightarrow{u_2} \in E$

$\alpha_1\overrightarrow{w_1} + \alpha_2\overrightarrow{w_2} = \alpha_1 T(\overrightarrow{u_1}) + \alpha_2 T(\overrightarrow{u_2}) = T(\alpha_1\overrightarrow{u_1} + \alpha_2\overrightarrow{u_2}) = T(\overrightarrow{u})$

where $\overrightarrow{u} = \alpha_1\overrightarrow{u_1} + \alpha_2\overrightarrow{u_2} \in E$

ie $\alpha_1\overrightarrow{w_1} + \alpha_2\overrightarrow{w_2} \in T(E)$

(c)

$$\begin{gathered}
T^{-1}(F) = \{\overrightarrow{u} \in V \mid T(\overrightarrow{u}) \in F\} \\
\text{Let } \overrightarrow{u_1}, \overrightarrow{u_2} \in T^{-1}(F)\ \alpha_1, \alpha_2 \in \mathbb{R} \\
T(\overrightarrow{u_1}) \in F, T(\overrightarrow{u_2}) \in F \\
\alpha_1\overrightarrow{u_1} + \alpha_2\overrightarrow{u_2} \in T^{-1}(F) \\
T(\alpha_1\overrightarrow{u_1} + \alpha_2\overrightarrow{u_2}) = \alpha_1 \underbrace{T(\overrightarrow{u_1})}_{\in F} + \alpha_2 \underbrace{T(\overrightarrow{u_2})}_{\in F} \in F \\
\alpha_1\overrightarrow{u_1} + \alpha_2\overrightarrow{u_2} \in T^{-1}(F)
\end{gathered} \tag{16}$$

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(From someone else's notes)

11.1 Linear Transformations

Proposition

$T : V \to W$ is a linear transformation

1. If E is a subspace of V then $T(E) = \{T(\overrightarrow{u})$ where $\overrightarrow{u} \in E\}$ is a subspace of W.
2. If F is a subspace of W then $T^{-1}(F) = \{\overrightarrow{u} \in V$ s.t. $T(\overrightarrow{u}) \in F\}$ is a subspace of V

Examples

1.

$$T : \mathbb{R}^3 \to \mathbb{R}^2$$
$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ z \end{pmatrix}$$
$$= \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{2\times 3} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{3\times 1} \tag{17}$$

will be a linear transformation because it can be written in this format (projection onto xz plane)

$$E = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \text{ (xy plane)}$$

if $\overrightarrow{u} \in E$ then $\overrightarrow{u} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$ and $T(\overrightarrow{u}) = \begin{bmatrix} a \\ 0 \end{bmatrix}$

$T(E) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \to x$ axis is in $\mathbb{R}^2$

2. Let $F = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

$$
\begin{aligned}
T^{-1}(F) &= \left\{ \overrightarrow{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ such that } T(\overrightarrow{u}) \in F \right\} \\
T(\overrightarrow{u}) &= \begin{pmatrix} x \\ z \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad t \in \mathbb{R} \\
& x = t, z = t, y \text{ is arbitrary } \rightarrow y = s \\
T^{-1}(F) &= \left\{ \overrightarrow{u} = \begin{pmatrix} t \\ s \\ t \end{pmatrix} \mid t, s \in \mathbb{R} \right\} \\
T^{-1}(F) &= \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}
\end{aligned}
\tag{18}
$$

$T^{-1}(F)$ is a subspace, $T^{-1}(F)$ cannot be empty, $\overrightarrow{0} \in T^{-1}(F)$

Particular Cases: $Ker(T)$ and $Im(T)$

Let $T : V \rightarrow W$ be a linear transformation

1. $E = V$ is a trivial subspace of V
   From previous proposition, $T(V)$ is a subspace of W. It is called the image of V through T, denoted $Im(T)$
2. $F = \left\{ \overrightarrow{0_W} \right\}$ is also a trivial subspace of W. Using the previous proposition $T^{-1}\left(\left\{ \overrightarrow{0_W} \right\}\right)$ is a subspace of V called the kernel of T, denoted $Ker(T)$.
   $Ker(T) = \left\{ \overrightarrow{u} \in V \text{ s.t. } T(\overrightarrow{u}) = \overrightarrow{0_W} \right\}$

Remark

$T : V \rightarrow W$ and $\{\overrightarrow{u_1}, \overrightarrow{u_2}, ..., \overrightarrow{u_n}, ...\}$ is a spanning set of V

then $\{T(\overrightarrow{u_1}), T(\overrightarrow{u_2}), ..., T(\overrightarrow{u_n}), ...\}$ is a spanning set of $T(V) = Im(T)$

Proof

Let $\overrightarrow{w} \in Im(T) \, (\equiv T(V))$, then $\exists \overrightarrow{u} \in V$ s.t. $\overrightarrow{w} = T(\overrightarrow{u}), \overrightarrow{u} = \sum_{i=1}^{n} \alpha_i \overrightarrow{u_i}, \overrightarrow{w} = T(\overrightarrow{u}) = T(\sum_{i=1}^{n} \alpha_i \overrightarrow{u_i}) = \sum_{i=1}^{n} \alpha_i T(\overrightarrow{u_i})$ (Linear combination of $T(\overrightarrow{u_i})$)

Examples

1. $T : V = \mathbb{R}^p \rightarrow W = \mathbb{R}^q$
   $T(X) = AX$

A spanning set of $Im(T)$ is given by $T(\overrightarrow{u_i})$ where $\overrightarrow{u_i} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{p \times 1} \leftarrow i^{th}$ position

$T(\overrightarrow{u_i}) = A\overrightarrow{u_i} = i^{th}$ column of A
$Im(T) = Col(A)$
$Ker(T) = \{x \mid AX = 0\} = Null(A)$

2. $T : V = P_2 \to W = \mathbb{R}$
   $T(p) = \int_0^1 p(x)dx$ is this a linear transformation?

   T is a linear transformation
   $\{1, , x^2\}$ is a spanning set of V
   $p_1(x) = 1, p_2(x) = x, p_3(x) = x^2$
   $T(p_1) = 1, T(p_2) = \frac{1}{2}, T(p_3) = \frac{1}{3} \to$ fractions came from doing transformations

$$\begin{aligned} Im(T) &= \text{span}\left\{1, \frac{1}{2}, \frac{1}{3}\right\} \subseteq \mathbb{R} \\ &= \text{span}\{1\} = \mathbb{R} \end{aligned} \tag{19}$$

   T is onto because the whole W is covered by $Im(T)$

   $Ker(T) = \left\{p \in P_2 \mid \int_0^1 p(x)dx = 0\right\}$
   $p(x) = ax^2 + bx + c \quad \int_0^1 p(x)dx \Rightarrow \frac{a}{3} + \frac{b}{2} - c = 0$
   $Ker(T) = \text{span}\left\{x - \frac{1}{2}, x^2 - \frac{1}{3}\right\}$
   $\frac{a}{3} + \frac{b}{2} + c = 0 \quad c = -\frac{a}{3} - \frac{b}{2}$
   $p(x) = ax^2 + bx - \frac{a}{3} - \frac{b}{2} = a(x^2 - \frac{1}{3}) + b(x - \frac{1}{2})$

3. $T : V = M_{2\times2} \to W = M_{2\times2}$
   $T(\underbrace{M}_{2\times2}) = \underbrace{A}_{2\times2}\underbrace{M}_{2\times2}$ where $A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$
   Is this a linear transformation? Yes
   (if $M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, AM = \begin{bmatrix} t\begin{bmatrix} 1 \\ 1 \end{bmatrix}, s\begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{bmatrix}$)

What is the basis of $M_{2\times2}$? $E_1 = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, E_2 = \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, E_3 = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}, E_4 = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$
$\{E_1, E_2, E_3, E_4\}$ is a basis of $M_{2\times2}$
$T(E_1) = \begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}, T(E_2) = \begin{bmatrix}2 & 0\\2 & 0\end{bmatrix}, T(E_3) = \begin{bmatrix}0 & 2\\0 & 2\end{bmatrix}, T(E_4) = \begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}$
$Im(T) = \text{span}\left\{\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}\right\}$ because clearly $T(E_2)$ is twice $T(E_1)$ & $T(E_3)$ is twice $T(E_4)$

$Ker(T) = \{M_{2\times2} \mid AM = 0\}$
$M = [x_1 \mid x_2] \Rightarrow AM = [Ax_1 \mid Ax_2] = 0$
$Ax_1 = 0 \quad x_1 = t\,[2 - 1]\, t \in \mathbb{R}$
$Ax_2 = 0 \quad x_2 = s\,[2 - 1]\, s \in \mathbb{R}$

$$M = \begin{bmatrix}t & s\\-t & -s\end{bmatrix} \quad t, s \in \mathbb{R}$$

$$Ker(T) = \text{span}\left\{\begin{pmatrix}2 & 0\\-1 & 0\end{pmatrix}, \begin{pmatrix}0 & 2\\0 & -1\end{pmatrix}\right\}$$

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Definition
(one-to-one and onto linear transformations)

1. $T : V \to W$, is said to be one-to-one (or injective) if, whenever $T(\overrightarrow{u_1}) = T(\overrightarrow{u_2})$, we have $\overrightarrow{u_1} = \overrightarrow{u_2}$
2. $T : V \to W$ is said to be onto (or surjective) if $W = J?m(T)$

Remark
$T : V \to W$ is onto iff there is a spanning set $\{\overrightarrow{u_1}, \overrightarrow{u_2}, ..., \overrightarrow{u_n}, ...\}$ of V such that $\{T(\overrightarrow{u_1}), T(\overrightarrow{u_2}), ..., T(\overrightarrow{u_n}), ...\}$ is a spanning set of W
Proposition
$T : V \to W$, linear transformation is one-to-one iff $Ker(T) = \left\{\overrightarrow{0_V}\right\}$
Proof

($\Rightarrow$) Assume T is one-to-one
Recall $Ker(T) = \left\{ \overrightarrow{u} \in v \mid T(\overrightarrow{u}) = \overrightarrow{0_W} \right\}$
Let $\overrightarrow{u} \in Ker(T), T(\overrightarrow{u}) = \overrightarrow{0_W} = T(\overrightarrow{0_V})$
$\because T$ is one-to-one, $\quad \therefore \overrightarrow{u} = \overrightarrow{0_V}$

($\Leftarrow$) Assume that $Ker(T) = \left\{ \overrightarrow{0_V} \right\}$
Let us prove that T is one-to-one
Let $\overrightarrow{u_1}, \overrightarrow{u_2} \in V$, such that

$$\begin{aligned} T(\overrightarrow{u_1}) &= T(\overrightarrow{u_2}) \Rightarrow \\ T(\overrightarrow{u_1} - \overrightarrow{u_2}) &= \overrightarrow{0_W} \\ \text{ie } \overrightarrow{u_1} - \overrightarrow{u_2} \in Ker(T) &= \left\{ \overrightarrow{0_V} \right\} \\ \overrightarrow{u_1} - \overrightarrow{u_2} &= \overrightarrow{0_V} \\ \text{ie } \overrightarrow{u_1} &= \overrightarrow{u_2} \end{aligned} \tag{20}$$

Examples

1. $T : P_2 \to P_3$

$$\begin{aligned} T(p)(x) &= \int_0^x p(t)dt \\ Ker(T) &= \{p \mid T(p) = 0\} \\ T(p)(x) &= 0 && \forall x \\ \frac{d}{dx}(T(p)(x)) &= 0 && \text{ie } p(x) = 0 \quad \forall x \text{ (By Fundamental theorem of calculus)} \\ Ker(T) &= \{0\} && \text{ie } T \text{ is one-to-one} \end{aligned} \tag{21}$$

   $\because T(p)(0) = 0 \quad \forall p \in P_2$
   $\therefore$ the polynomial $f(x) = 1 \quad \forall x$ does not belong to $Im(T)$, ie $Im(T) \neq P_3$ Exercise

   Prove that $Im(T) = \{p \in P_3 \mid p(0) = 0\}$

2. $T : \mathbb{R}^n \to \mathbb{R}^m$
   $T(X) = \underbrace{A}_{m \times n} X$

   (a) T is one-to-one iff the homogeneous system $AX = 0$ has a unique solution
   ie $Rank(A) = n$

(b) T is onto iff $Col(A) = \mathbb{R}^m$
ie $Rank(A) = dim(Col(A)) = m$

Remark

$T : V \to W$
Let $\overrightarrow{w}$ be a fixed vector in W
Solving the equation $T\overrightarrow{u} = \overrightarrow{w}$
The set of all solutions is $T^{-1}(\{\overrightarrow{w}\})$

(a) $T^{-1}(\{\overrightarrow{w}\})$ can be empty (No solution)

(b) $T^{-1}(\{\overrightarrow{w}\})$ can have only one vector if
$\overrightarrow{w} \in Im(T)$ and T is one-to-one item $T^{-1}(\{\overrightarrow{w}\})$ has infinitely many vectors when
$\overrightarrow{w} \in Im(T)$ and $Ker(T) \neq \left\{\overrightarrow{0_V}\right\}$

12.1 Isomorphism

Definition
A linear transformation $T : V \to W$ is said to be an isomorphism if T is one-to-one and onto
Examples

1. $T : \mathbb{R}^n \to \mathbb{R}^n$
   $T(X) = \underbrace{A}_{n\times n} X$
   T is an isomorphism iff $Rank(A) = n$
2. $T : M_{n\times m} \to M_{m\times n}$
   $T(A) = A^T$
   Recall $\underbrace{A}_{n\times m} \in Ker(T) \Leftrightarrow T(A) = A^T = \underbrace{0}_{m\times n}$
   $A = (A^T)^T = (\underbrace{0}_{m\times n})^T = \underbrace{0}_{m\times n}$
   T is one-to-one
   Let $B \in M_{m\times n}$
   T is onto

   T is an isomorphism

3. The coordinate map
   If V is a vector space such that $dim(V) = n$ and $\{\overrightarrow{u_1}, \overrightarrow{u_2}, ..., \overrightarrow{u_n}\} = B$ is a basis of V
   The map $T : V \to \mathbb{R}^n$
   $T(\overrightarrow{u}) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \leftarrow$ coordinates of $\overrightarrow{u}$ relative to B
   is an isomorphism

<u>Proposition</u>
Let $T : V \to W$ be an isomorphism
The inverse transformation $T^{-1} : W \to V$ is a linear transformation, and is also an isomorphism
<u>Proof</u>
$T \circ T^{-1}(\overrightarrow{w}) = \overrightarrow{w} \quad \overrightarrow{w} \in W$
$T^{-1} \circ T(\overrightarrow{v}) = \overrightarrow{v} \quad \overrightarrow{v} \in V$
Let $\overrightarrow{w_1}, \overrightarrow{w_2} \in W$, $\alpha_1, \alpha_2 \in \mathbb{R}$

$$\begin{aligned} T^{-1}(\alpha_1\overrightarrow{w_1} + \alpha_2\overrightarrow{w_2}) &\stackrel{?}{=} \alpha_1 T^{-1}(\overrightarrow{w_1}) + \alpha_2 T^{-1}(\overrightarrow{w_2}) \\ T(T^{-1}(\alpha_1\overrightarrow{w_1} + \alpha_2\overrightarrow{w_2})) &= \alpha_1\overrightarrow{w_1} + \alpha_2\overrightarrow{w_2} \\ T(\alpha_1 T^{-1}(\overrightarrow{w_1}) + \alpha_2 T^{-1}(\overrightarrow{w_2})) &= \alpha_1 T(T^{-1}(\overrightarrow{w_1})) + \alpha_2 T(T^{-1}(\overrightarrow{w_2})) \\ &= \alpha_1\overrightarrow{w_1} + \alpha_2\overrightarrow{w_2} \end{aligned} \tag{22}$$

$\because T$ is one-to-one
$T^{-1}(\alpha_1\overrightarrow{w_1} + \alpha_2\overrightarrow{w_2}) = \alpha_1 T^{-1}(\overrightarrow{w_1}) + \alpha_2 T^{-1}(\overrightarrow{w_2})$
$Im(T^{-1}) = V$
$\because \forall \overrightarrow{v} \in V$
$\overrightarrow{v} = T^{-1}(T(\overrightarrow{v}))$ ie $\overrightarrow{v} \in Im(T^{-1})$
$\overrightarrow{w} \in Ker(T^{-1}) \quad T^{-1}(\overrightarrow{w}) = \overrightarrow{0_V}$
$\overrightarrow{w} = T(T^{-1}(\overrightarrow{w})) = T(\overrightarrow{0_V}) = \overrightarrow{0_W}$
$Ker(T^{-1}) = \left\{\overrightarrow{0_W}\right\} \quad T^{-1}$ is one-to-one
T^{-1} is also an isomorphism
<u>Exercise</u>
Let $T : V \to W$ be an isomorphism and $B = \{\overrightarrow{u_1}, \overrightarrow{u_2}, ..., \overrightarrow{u_n}\}$ be a basis of V
Let $\overrightarrow{w_i} = T(\overrightarrow{u_i})$
Prove that $\{\overrightarrow{w_1}, \overrightarrow{w_2}, ..., \overrightarrow{w_n}\}$ is a basis of W

$(\therefore dim(V) = dim(W))$

Dimension Theorem

(Generalization of the Rank Theorem)

$\mathbb{R}^m \rightarrow \mathbb{R}^n \quad \underbrace{A}_{n\times m}$

$$\underbrace{Real(A)}_{dim(col(A))=dim(Im(T))} + \underbrace{dim(Null(A))}_{dim(Ker(T))} = m$$

Theorem

Let $T : V\mathbb{R}W$ be a linear transformation.

Assume $dim(V)$ is finite

Then $dim(V) = dim(Ker(T)) + dim(Im(T))$

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\\TODO

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Midterm is on Chapter 4 & 5, and has 4 questions. Rooms will be announced tomorrow; try to get there 10 min early.

Examples

1. Given E, F are subspaces of V, prove that $E \oplus F \subseteq V$.

   We already know that $E + F \subseteq V$, so we just need to show that $E \cap F = \left\{\overrightarrow{0_v}\right\}$

   To prove equality, show that $dim(V) = dim(E) + dim(F)$

2. If U is a subspace of V, W is a subspace of V, and $U \cup W$ is a subspace of V, prove that $U \subseteq W$ or $W \subseteq U$

   If $U \nsubseteq W$ and $W \nsubseteq U$, take $\overrightarrow{u} \in U$ where $\overrightarrow{u} \notin W$, and $\overrightarrow{w} \in W$ where $\overrightarrow{w} \notin U$, then $\overrightarrow{u} + \overrightarrow{w} \notin U \cup W$

3. $T; V \rightarrow W$

   (a) E is a subspace of V
   $dim(T(E)) = dim(E) - dim(Ker(T) \cap E)$

Define $T_1 : E \to T(E)$

$$\begin{aligned} dim(E) &= \underbrace{dim(Im(T_1))}_{=dim(T(E))} + \underbrace{dim(Ker(T_1))}_{=dim(E\cap Ker(T))} \\ \overrightarrow{u} \in Ker(T_1) &\Leftrightarrow \overrightarrow{u} \in E \text{ and } T(\overrightarrow{u}) = \overrightarrow{0_w} \\ &\Leftrightarrow \overrightarrow{u} \in E \text{ and } \overrightarrow{u} \in Ker(T) \\ Ker(T_1) &= Ker(T) \cap E \end{aligned} \tag{23}$$

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15 2018/03/13

(Copied from someone else's notes)

15.1 Matrix Representation of Linear Transformation

$T : V \to V, dim(V) = n$

If $B = \{\overrightarrow{u_1}, \overrightarrow{u_2}, ..., \overrightarrow{u_n}\}$ is a basis of V,

there exists a (unique) $n \times n$ matrix denoted $[T]_B$ such that $\forall u \in V$

$\underbrace{[T(\overrightarrow{u})]_B}_{note1} = [T]_B \underbrace{[\overrightarrow{u}]_B}_{note2}$

note 1: coordinates of $T(\overrightarrow{u})$ relative to B

note 2: coordinates of $\overrightarrow{u}$ relative to B

Remark

Given that $B = \{\overrightarrow{u_1}, \overrightarrow{u_2}, ..., \overrightarrow{u_n}\}$, the i^{th} column of $[T]_B$ is $[T(u_i)]_B$

Properties

1. If T_1 and T_2 are linear transformations from V into V: $[T_1 + T_2]_B = [T_1]_B + [T_2]_B$

$$\begin{aligned} \because \quad [(T_1 + T_2)(u)]_B &= [T_1(u) + T_2(u)]_B \\ &= [T_1(u)]_B + [T_2(u)]_B \\ &= ([T_1]_B + [T_2]_B)\,[u]_B \\ \therefore \quad [T_1 + T_2]_B &= [T_1]_B + [T_2]_B \end{aligned} \tag{24}$$

2. If $T : V \to V$ is a linear transformation and $\alpha \in \mathbb{R}$, $[\alpha T]_B = \alpha[T]_B$

3. Let $T_1 : V \to V$, $T_2 : V \to V$ be 2 linear transformations. $[T_1 \circ T_2]_B = [T_1]_B[T_2]_B$

$$\begin{aligned}[T_1 \circ T_2(u)] &= [T_1(T_2(u))]_B \\ &= [T_1]_B[T_2(u)]_B \\ &= [T_1]_B[T_2]_B[u]_B\end{aligned} \tag{25}$$

4. $T : V \to V$ is an isomorphism iff $[T]_B$ is invertible. Moreover, $[T^{-1}]_B = ([T]_B)^{-1}$

15.2 Change of Basis

$T : V \to V dim(V) = n$
$B = \{\overrightarrow{u_1}, \overrightarrow{u_2}, ..., \overrightarrow{u_n}\}$ and $S = \{\overrightarrow{v_1}, \overrightarrow{v_2}, ..., \overrightarrow{v_n}\}$

Let P be the $n \times n$ matrix such that every i^{th} column is $[v_i]_B$
If $\overrightarrow{u} \in V$, then $[u]_B = P[u]_S$

$\overrightarrow{u} = \sum x_i v_i \quad [u]_S = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $\qquad [u]_B = \sum_i x_i[v_i]_B = P \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = P[u]_S$

P is the transition matrix from S to B

Let $u \in V$

$$\begin{aligned}[T(u)]_S &= P^{-1}[T(u)]_B \\ &= P^{-1}[T]_B[u]_B \\ &= P^{-1}[T_B]P[u]_S \\ \therefore \quad [T]_S &= P^{-1}[T]_B P\end{aligned} \tag{26}$$

Example

$$
\begin{aligned}
V &= P_2 \\
T(p)(n) &= xp'(n) \\
B &= \left\{ \underbrace{1}_{p_1}, \underbrace{x}_{p_2}, \underbrace{x^2}_{p_3} \right\} \\
S &= \left\{ \underbrace{1+x}_{q_1}, \underbrace{2x-1}_{q_2}, \underbrace{x^2+x}_{q_3} \right\} \\
T(p_1)(x) &= 0 \\
T(p_2)(x) &= x \\
T(p_3)(x) &= 2x^2 \\
[T]_B &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\
Null([T]_B) &= \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} = Ker(T) \\
Ker(T) &= \{tp_1, t \in \mathbb{R}\} = \text{span}\,\{p_1\} \\
T(q_1)(x) &= x = \frac{1}{3}(q_1 + q_2) \\
T(q_2)(x) &= 2x = \frac{2}{3}(q_1 + q_2) \\
T(q_3)(x) &= 2x^2 + x = 2(x^2 + x) - x = 2q_3 - \frac{1}{3}(q_1 + q_2) \\
[T]_B &= \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 2 \end{bmatrix}
\end{aligned} \tag{27}
$$

Using the formula $[T]_S = P^{-1}[T]_B P$

$P = \begin{bmatrix} 1 & =1 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ $P^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix}$

15.2.1 Generalization

$T : V \to W$ is a linear transformation
$dim(V) = n, B = \{\overrightarrow{u_1}, \overrightarrow{u_2}, ..., \overrightarrow{u_n}\}$ is a basis of V
$dim(W) = m, S = \{\overrightarrow{v_1}, \overrightarrow{v_2}, ..., \overrightarrow{v_n}\}$ be a basis of W
There exists a unique $m \times n$ matrix
$[T]_{B,S}$ is called the matrix of T relative to the bases B and S such that $\forall u \in V$
$\underbrace{[T(u)]_S}_{m\times 1} = \underbrace{[T]_{B,S}}_{m\times n}\underbrace{[u]_B}_{n\times 1}$
Remark
The j^{th} column of $[T]_{B,S}$ is $[T(u_j)]_S$
Example

$$\begin{aligned}
T &: P_3 \to P_2 \\
T(p(n) &= p'(n) \\
B &= \left\{1, x, x^2, x^3\right\} \\
S &= \left\{\underbrace{1+x}_{q_1}, \underbrace{2x-1}_{q_2}, \underbrace{x^2+x}_{q_3}\right\} \\
3x^2 &= 3[(x^2+x)-x] \\
&= 3q_3 - 3x \\
&= 3q_3 - q_1 - q_2 \\
[T]_{BS} &= \begin{bmatrix} 0 & \frac{2}{3} & \frac{2}{3} & -1 \\ 0 & -\frac{1}{3} & \frac{2}{3} & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix}
\end{aligned} \tag{28}$$

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Generalization

$$T : V \to W$$

- B is a basis of V, $dim(V) = n$
- S is a basis of W, $dim(W) = m$

Then there exists a unique $m \times n$ matrix
$[T]_{B,S}$ such that whenever $u \in V$, $[T(u)]_S = [T]_{B,S}\,[u]_B$

<u>Proposition</u>

Let V_1, V_2, V_3 be 3 vector spaces. $dim(V_i) = n_i$ and B_i is a basis of V_i $(i = 1, 2, 3)$
Let $F : V_1 \to V_2$ and $G : V_2 \to V_3$ be a linear transformation.
$G \circ F : V_1 \to V_3$ is a linear transformation such that

$$\underbrace{[G \circ F]_{B_1,B_3}}_{n_3 \times n_1} = \underbrace{[G]_{B_2,B_3}}_{n_3 \times n_2}\,\underbrace{[F]_{B_1,B_2}}_{n_2 \times n_1}$$

Application

$V_B \xrightarrow{T} V_B \quad [T]_B$

$\uparrow Id \qquad \downarrow Id$

$V_S \xrightarrow{T} V_S \quad [T]_S$

$$\begin{aligned} T &= Id \circ T \circ Id \quad \Big|^{B=\{\overrightarrow{u_1},\overrightarrow{u_2},\ldots,\overrightarrow{u_n}\}}_{S=\{\overrightarrow{v_1},\overrightarrow{v_2},\ldots,\overrightarrow{v_n}\}} \\ [T]_S &= [Id]_{B,S}\,[T]_B\,[Id]_{S,B} \end{aligned} \tag{29}$$

Note that the column of $[Id]_{B,S}$ is $[u_i]_S$

Therefore $[Id]_{B,S} = P$, the transition matrix from B to S.

$[T]_S = P\,[T]_B\,P^{-1}$

16.1 Similar Matrices

Two $n \times n$ matrices A, B are said to be similar if there exists an invertible matrix P, such that $A = PBP^{-1}$

Remark

1. If A and B are similar, $det(A) = det(B)$
2. tr(A) = tr(B)

Discussed and got back midterms

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17.1 Inner Product

Review: Dot Product

$u, v \in \mathbb{R}^n$

$u \cdot v = \sum_{i=1}^{n} u_i v_i$ where $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

Properties

$$\begin{aligned} u \cdot v &= v \cdot u \\ (u_1 + u_2) \cdot v &= u_1 \cdot v + u_2 \cdot v \\ (\alpha u) \cdot v &= \alpha(u \cdot v) \\ u \cdot u &= \sum_{i=1}^{n} u_i^2 \geq 0 \end{aligned} \tag{30}$$

Cauchy-Schwarz Inequality

$|u \cdot v| \leq \sqrt{u \cdot u}\sqrt{v \cdot v} = ||u||\,||v||$

If $u \neq 0$ and $v \neq 0$, then $\frac{|u \cdot v|}{||u||||v||} \leq 1$ ie $-1 \leq \frac{|u \cdot v|}{||u||||v||} \leq 1$

$\frac{|u \cdot v|}{||u||||v||} = cos(\theta)$ where $\theta \in [0, \pi]$

θ is called the angle between u and v

Definition (Inner Product)

Let V be a vector space.

An inner product on V is a function denoted $\langle , \rangle : V \times V \to \mathbb{R}$.

(It associates to any pair $(u, v) \in V \times V$ as a number denoted $\langle u, v \rangle$)

Properties

1. $\langle u, v \rangle = \langle v, u \rangle$ (Symmetry)
2. Whenever $u_1, u_2, v \in V$ $\alpha_1 \alpha_2 \in \mathbb{R}$:
   $\langle \alpha_1 u_1 + \alpha_2 u_2, v \rangle = \alpha_1 \langle u_1, v \rangle + \alpha_2 \langle u_2, v \rangle$
3. $\langle u, u \rangle \geq 0$ and $\langle u, u \rangle = 0$ iff $u = 0_v$

Examples

1. $V = \mathbb{R}^n$
   $\langle u, v \rangle = u \cdot v = \underbrace{\begin{bmatrix} u \end{bmatrix}^T}_{1 \times n} \underbrace{\begin{bmatrix} v \end{bmatrix}}_{n \times 1}$
2. $V = \mathbb{R}^2$ $u_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ $u_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$

$$\begin{aligned}
\langle u_1, u_2 \rangle &= x_1x_2 - y_1y_2 \\
&= \begin{bmatrix} x_1 & y_1 \end{bmatrix} \begin{bmatrix} x_2 \\ -y_2 \end{bmatrix} \\
&= \begin{bmatrix} x_1 & y_1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{A} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \\
&= \begin{bmatrix} u_1 \end{bmatrix}^T A \begin{bmatrix} u_2 \end{bmatrix}
\end{aligned} \tag{31}$$

(a)

$$\begin{aligned}
\langle u_2, u_1 \rangle &= \begin{bmatrix} u_2 \end{bmatrix}^T A \begin{bmatrix} u_1 \end{bmatrix} \\
\langle u_2, u_1 \rangle &= \underbrace{\begin{bmatrix} u_2 \end{bmatrix}^T A \begin{bmatrix} u_1 \end{bmatrix}}_{1\times 1} \\
&= \left(\begin{bmatrix} u_2 \end{bmatrix}^T A \begin{bmatrix} u_1 \end{bmatrix} \right)^T \\
&= \begin{bmatrix} u_1 \end{bmatrix}^T A^T \begin{bmatrix} u_2 \end{bmatrix} \qquad (A = A^T) \\
&= \begin{bmatrix} u_1 \end{bmatrix}^T A \begin{bmatrix} u_2 \end{bmatrix} \\
&= \langle u_1, u_2 \rangle
\end{aligned} \tag{32}$$

(b)

$$\begin{aligned}
\langle \alpha_1 u_1 + \alpha_2 u_2, v \rangle &= \begin{bmatrix} \alpha_1 u_1 + \alpha_2 u_2 \end{bmatrix}^T A \begin{bmatrix} v \end{bmatrix} \\
&= \left(\alpha_1 \begin{bmatrix} u_1 \end{bmatrix}^T + \alpha_2 \begin{bmatrix} u_2 \end{bmatrix}^T \right) A \begin{bmatrix} v \end{bmatrix} \\
&= \alpha_1 \langle u_1, v \rangle + \alpha_2 \langle u_2, v \rangle
\end{aligned} \tag{33}$$

(c)

$$\begin{aligned}
\langle u, u \rangle &= \begin{bmatrix} u \end{bmatrix}^T A \begin{bmatrix} u \end{bmatrix} \\
&= x^2 - y^2 \qquad \text{if } \begin{bmatrix} u \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}
\end{aligned} \tag{34}$$

If $\begin{bmatrix} u \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then $\langle u, u \rangle = -1 < 0$

Therefore, $\langle , \rangle$ is not an inner product on $V = \mathbb{R}^2$

3. $V = \mathbb{R}^2$

$\langle u, v\rangle = \begin{bmatrix} u \end{bmatrix}^T A \begin{bmatrix} v \end{bmatrix}$ where $A = \begin{bmatrix} 2 & -2 \\ -2 & 6 \end{bmatrix}$

$u_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ $u_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$

$\langle u_1, u_2\rangle = 2x_1x_2 - 2(x_1y_2 + x_2y_1) + 6y_1y_2$

Since $A = A^T$, $\langle u, v\rangle = \langle v, u\rangle$

It is clear that $\langle \alpha_1 u_1 + \alpha_2 u_2, v\rangle = \alpha_1\langle u_1, v\rangle + \alpha_2\langle u_2, v\rangle$

$$\begin{aligned}
\langle u, u\rangle &= \begin{bmatrix} u \end{bmatrix}^T A \begin{bmatrix} u \end{bmatrix} \qquad \text{if } \begin{bmatrix} u \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \\
\langle u, u\rangle &= 2x^2 - 4xy + 6y^2 \\
&= 2(x^2 - 2x6) + 6y^2 \\
&= 2((x-y)^2 - y^2) + 6y^2 \\
&= 2(x-y)^2 + 4y^2 \geq 0
\end{aligned} \tag{35}$$

$$\text{Moreover,} \langle u, u\rangle = 0 \Leftrightarrow \begin{cases} x - y = 0 \\ y = 0 \end{cases} \quad \text{ie } \begin{bmatrix} u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{36}$$

17.2 Diagonalization of A

(Based on example 3 above for $A = \begin{bmatrix} 2 & -2 \\ -2 & 6 \end{bmatrix}$)

1. Characteristic Polynomial

$$\begin{aligned}
P_A &= det(A - \lambda I_2) \\
&= det \begin{bmatrix} 2-\lambda & -2 \\ -2 & 6-\lambda \end{bmatrix} \\
&= (\lambda - 2)(\lambda - 6) - 4 \\
&= \lambda^2 - 8\lambda + 12 - 4 \\
&= \lambda^2 - 8\lambda + 8
\end{aligned} \tag{37}$$

2. Eigenvalues

$$\begin{aligned}\lambda_1 &= 4 + 2\sqrt{2}\\ \lambda_2 &= 4 - 2\sqrt{2}\end{aligned} \tag{38}$$

3. Eigenvectors

$$A - \lambda_1 I_2 = \begin{pmatrix} -2+2\sqrt{2} & -2 \\ -2 & 2+2\sqrt{2} \end{pmatrix}: \quad x_1 = \begin{bmatrix} 1 \\ -1+\sqrt{2} \end{bmatrix} \text{ is an eigenvector}$$

$$A - \lambda_2 I_2 = \begin{pmatrix} -2-2\sqrt{2} & -2 \\ -2 & 2-2\sqrt{2} \end{pmatrix}: \quad x_2 = \begin{bmatrix} 1 \\ -1-\sqrt{2} \end{bmatrix} \text{ is an eigenvector}$$

$$x_1 \cdot x_2 = 1 + \left((-1)^2 - (\sqrt{2})^2\right)^2 = 0$$

4. Diagonalization

$$A = PDP^{-1} = \begin{bmatrix} 1 & 1 \\ -1+\sqrt{2} & -1-\sqrt{2} \end{bmatrix} \begin{bmatrix} 4-2\sqrt{2} & 0 \\ 0 & 4+2\sqrt{2} \end{bmatrix} \left(\frac{1}{-2\sqrt{2}} \begin{bmatrix} -1-\sqrt{2} & -1 \\ 1-\sqrt{2} & 1 \end{bmatrix} \right)$$

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$\langle u, v \rangle = \begin{bmatrix} u \end{bmatrix}^T A \begin{bmatrix} v \end{bmatrix}$

$\langle u, v \rangle = \langle v, u \rangle$ for this property to hold, we need $A = A^T$; ie A must be symmetric

$\langle u, v \rangle$ is clearly linear in u.

The last property: $\langle u, u \rangle \geq 0$ and $\langle u, u \rangle = 0 \Leftrightarrow u = 0$

Definition

A symmetric matrix A such that $\begin{bmatrix} u \end{bmatrix}^T A \begin{bmatrix} u \end{bmatrix} \geq 0 \ \forall u \in \mathbb{R}^n$ and $\begin{bmatrix} u \end{bmatrix}^T A \begin{bmatrix} u \end{bmatrix} = 0$ only for $u = 0$ is called a positive definite matrix

Example

I_n is an $n \times n$ positive definite matrix

Proposition

If A is positive definite, then $\langle u, v \rangle = \begin{bmatrix} u \end{bmatrix}^T A \begin{bmatrix} v \end{bmatrix}$ defines an inner product on $\mathbb{R}^n$

Theorem

If A is a symmetric matrix then A is diagonalizable. Moreover, there exists a matrix Q such that $Q^{-1} = Q^T$ and $A = QDQ^T$ where D is a diagonal matrix.

Remark

An $n \times n$ matrix, such that $Q^{-1} = Q^T$ (or equivalently $QQ^T = Q^TQ = I_n$) is called an orthogonal matrix.
If x_i is the i^{th} column of Q, then $x_i^T x_i = 1$ and $x_i^T x_j = 0$ whenever $i \neq j$
Example
($n = 2$)
$A = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$

$$\begin{aligned} P_A(\lambda) &= det(A - \lambda I_2) \\ &= (\lambda - 5)^2 - 1 \\ &= (\lambda - 6)(\lambda - 4) \end{aligned} \tag{39}$$

$$\lambda_1 = 6$$
$$\lambda_2 = 4$$

- $\lambda_1 = 6 \quad A - \lambda_1 I_2 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$
  $A - \lambda_1 I_2)X = 0 \Leftrightarrow X = t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \; t$
- $\lambda_2 = 4 \quad A - \lambda_2 I_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$
  $A - \lambda_2 I_2)X = 0 \Leftrightarrow X = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$A = PDP^{-1}$ where $D = \begin{bmatrix} 6 & 0 \\ 0 & 4 \end{bmatrix}, P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$
Choose $Q = \begin{bmatrix} \frac{1}{\sqrt{}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ Q is an orthogonal matrix $A = QDQ^T$
Question
Is $\langle u, v \rangle = \begin{bmatrix} u \end{bmatrix}^T A \begin{bmatrix} v \end{bmatrix}$ an inner product in $\mathbb{R}^2$? Equivalently, is A a positive definite matrix?

$$\begin{aligned} \langle u, u \rangle &= \begin{bmatrix} u \end{bmatrix}^T A \begin{bmatrix} u \end{bmatrix} \\ &= \begin{bmatrix} u \end{bmatrix}^T QDQ^T \begin{bmatrix} u \end{bmatrix} \\ &= \left(Q^T \begin{bmatrix} u \end{bmatrix}\right)^T D \left(Q^T \begin{bmatrix} u \end{bmatrix}\right) \end{aligned} \tag{40}$$

Let S be the basis of $\mathbb{R}^2$ such that
$\begin{bmatrix}u\end{bmatrix}_S = Q^T \begin{bmatrix}u\end{bmatrix} \qquad S = \left\{ \begin{bmatrix}\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}\end{bmatrix}, \begin{bmatrix}-\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}\end{bmatrix} \right\}$
$\langle u, u\rangle = \begin{bmatrix}u\end{bmatrix}_S^T D \begin{bmatrix}u\end{bmatrix}_S \quad \text{if } \begin{bmatrix}u\end{bmatrix}_S = \begin{bmatrix}x_1 \\ x_2\end{bmatrix}$
$\langle u, u\rangle = \lambda_1 x_1^2 + \lambda_2 x_2^2$
This defines an inner product iff $\lambda_1 > 0$ and $\lambda_2 > 0$
$A = \begin{bmatrix}5 & 1 \\ 1 & 5\end{bmatrix}$
$u = \begin{bmatrix}x \\ y\end{bmatrix} \quad v = \begin{bmatrix}x' \\ y'\end{bmatrix}$

$$\begin{aligned}\langle u, v\rangle &= 5xx' + (xy' + x'y) + 5yy' \\ \langle u, u\rangle &= 5x^2 + 2xy + 5y^2 \\ &= 6x_1^2 + 4y_1^2\end{aligned} \tag{41}$$

<u>Exercise</u>
Let $A = \begin{bmatrix}-4 & -2 & 4 \\ -2 & -1 & 2 \\ 4 & 2 & -4\end{bmatrix}$

1. Find Q such that $A = QDQ^T$; Q is the orthogonal matrix, and D is the diagonal matrix
2. is $\langle u, v\rangle = \begin{bmatrix}u\end{bmatrix}^T A \begin{bmatrix}v\end{bmatrix}$ an inner product in $\mathbb{R}^3$?

<u>Other examples of inner product</u>

1. Let $V = P_n$
   $\langle p, q\rangle = \int_0^1 p(t)q(t)dt$
   (Example: $p(t) = t, q(t) = t - 1, \langle p, q\rangle = \int_0^1 t(t-1)dt = -\frac{1}{6}$)

   (a)
   $$\begin{aligned}\langle q, p\rangle &= \int_0^1 q(t)p(t)dt \\ &= \int_0^1 p(t)q(t)dt = \langle p, q\rangle\end{aligned} \tag{42}$$

(b)

$$\begin{aligned}\langle p_1 + p_2, q\rangle &= \int_0^1 (p_1(t) + p_2(t))q(t)dt \\ &= \int_0^1 p_1(t)q(t)dt + \int_0^1 p_2(t)q(t)dt \\ &= \langle p_1, q\rangle + \langle p_2, q\rangle\end{aligned} \tag{43}$$

(c)

$$\begin{aligned}\langle p, p\rangle = 0 &\Leftrightarrow \int_0^1 p^2(t)dt = 0 \\ &\Leftrightarrow p^2(t) = 0 \quad \forall t \in (0, 1) \\ &\Leftrightarrow p(t) = 0 \quad \forall t \in (0, 1)\end{aligned} \tag{44}$$

$x_i = \frac{1}{i} \quad i = 1, 2, 3, ..., n$

$x_i \in (0, 1) \quad P(x_i) = 0$

$p(t) = C(x - x_i)...(x - x_n)$

$p(x_{n+1}) = 0 \Leftrightarrow C(x_{n+1} - x_i)...(x_{n+1} - x_n) = 0 \Rightarrow C = 0 \therefore p = 0$

Let $\underbrace{\rho(t) > 0}_{\text{weight function}}$ on $(0, 1)$

$\langle p, q\rangle_\rho = \int_0^1 \rho(t)p(t)q(t)dt$

2. $V = M_{n\times n}$

$\langle A, B\rangle = tr(A^T B)$

(a)

$$\begin{aligned}\langle B, A\rangle &= tr(B^T A) \\ &= tr\left((B^T A)^T\right) \\ &= tr(A^T B) \\ &= \langle A, B\rangle\end{aligned} \tag{45}$$

(b)

$$\langle A, A\rangle = tr(A^T A)$$

$$A = (a_{ij}) \tag{46}$$

$$tr(A^T A) = \sum_i (A^T A)$$

Let $\underbrace{x_i}_{n\times 1}$ be the i^{th} column of A

$(A^T A)_{ii} = X_i^T x_i$

$tr(A^TA) = \sum_{i=1}^r x_i^T x_i \geq 0$

also

$$\begin{aligned} \langle A, A\rangle = 0 \Rightarrow \quad & x_i^T x_i = 0 \quad \forall i \\ ie \quad & x_i = \underbrace{0}_{n\times 1} \quad \forall i \\ ie \quad & A = \underbrace{0}_{n\times n} \end{aligned} \tag{47}$$

Cauchy-Schwarz Inequality

Proposition

Let $\langle , \rangle$ be an inner product on a vector space V

$\langle u, v\rangle\Big|^2 \leq \langle u, u\rangle\langle v, v\rangle$

Proof

Let $u, v \in V$ be fixed

Define $F(t) = \langle u + tv, u + tv\rangle$

Note that $F(t) \geq 0 \quad \forall t$

Also $F(t) = \langle u, u\rangle + 2t\langle u, v\rangle + t^2\langle v, v\rangle$

Therefore the discriminant $\Delta' = \langle u, v\rangle^2 - \langle u, u\rangle\langle v, v\rangle \leq 0$

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\\TODO

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Exercises

1. Assume that E, R are 2 subspaces of V such that $E \perp F$.
   Prove that $P_{E\oplus F} = P_E + P_F$
2. Does the above hold if E is not orthogonal to F?

21.1 More about Projections

21.1.1 $\mathbb{R}^n$ with the usual dot project

If $T, \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformations, let $\begin{bmatrix}T\end{bmatrix}$ be the standard matrix of T.

What are the condition(s) on $\begin{bmatrix}T\end{bmatrix}$ so that $\begin{bmatrix}T\end{bmatrix}$ is the standard matrix of an orthogonal projection onto a subspace E of $\mathbb{R}^n$?

If T is an orthogonal projection
$T^2(u) = T(\underbrace{T(u)}_{\in E}) = T(u), T^2 = T, \begin{bmatrix}T\end{bmatrix}^2 = \begin{bmatrix}T\end{bmatrix}$

$\mathbb{R}^n = Ker(T) \oplus Im(T) \qquad E = Im(T)$
If T is an orthogonal projection, we must also have $Ker(T) \perp Im(T)$
\\TODO add spans
$u = \begin{bmatrix}x \\ y\end{bmatrix} = (x-y)\begin{bmatrix}1 \\ 0\end{bmatrix} + y\begin{bmatrix}1 \\ 1\end{bmatrix}$
Let $\begin{bmatrix}T(u)\end{bmatrix} = \begin{bmatrix}y \\ y\end{bmatrix} = \begin{bmatrix}T\end{bmatrix}\begin{bmatrix}x \\ y\end{bmatrix} = \begin{bmatrix}0 & 1 \\ 0 & 1\end{bmatrix}\begin{bmatrix}x \\ y\end{bmatrix}$
$\begin{bmatrix}T\end{bmatrix}^2 = \begin{bmatrix}T\end{bmatrix}, Ker(T) = \text{span}\left\{\overrightarrow{i}\right\}, Im(T) = \text{span}\left\{\overrightarrow{i} + j\right\}$

For T to be an orthogonal projection on $Im(T)$, we must have $Im(T) \perp Ker(T)$
ie $\forall u, v \in \mathbb{R}^n$

$$\begin{aligned}
\langle T(u), v - T(v)\rangle &= 0 \\
\langle T(u), v\rangle &= \langle T(u), T(v)\rangle = \langle u, T(V)\rangle \\
\langle T(u), v\rangle &= \langle u, T(v)\rangle \\
\begin{bmatrix}T(u)\end{bmatrix}^T \begin{bmatrix}v\end{bmatrix} &= \langle u\rangle^T \langle T(v)\rangle \\
\begin{bmatrix}\begin{bmatrix}T\end{bmatrix}\begin{bmatrix}u\end{bmatrix}\end{bmatrix}^T \begin{bmatrix}v\end{bmatrix} &= \begin{bmatrix}u\end{bmatrix}^T \begin{bmatrix}T\end{bmatrix}\begin{bmatrix}v\end{bmatrix}
\end{aligned} \tag{48}$$

<u>Proposition</u>
Let P be an $n \times n$ matrix such that $P^2 = P$. P is the standard matrix of an orthogonal projection iff $P^T = P$
<u>Example</u>
From lecture 18:
$A = \begin{bmatrix}5 & 1 \\ 1 & 5\end{bmatrix} \qquad$ eigenvalues: $\lambda_1 = 6, \lambda_2 = 4$

$\lambda = 6 \Rightarrow E_6 = \text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}\right\}$

$\lambda = 4 \Rightarrow E_4 = \text{span}\left\{\begin{bmatrix}-1\\1\end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix}-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}\right\}$

$A = QDQ^T$ where $Q = \begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}, D = \begin{bmatrix}6 & 0\\0 & 4\end{bmatrix}$

Exercise

Prove that $A = 6P_{E_6} + 4P_{E_4}$

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\\TODO away

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$$\begin{gathered}(E+F)^\perp = E^\perp \cap F^\perp\\ \text{If } E \perp F \text{ then } P_{E\oplus F} = P_E + P_F\end{gathered} \tag{49}$$

Exercise

If $P_{E+F} = P_E + P_F$, is it necessary that $E \perp F$? E, F are subspaces of V.

$T : V \to V$ is an isomorphism iff $\underbrace{\begin{bmatrix}T\end{bmatrix}_B}_{n\times n}$ is invertible.

Let $\underbrace{X}_{n\times 1}$ be such that $\begin{bmatrix}T\end{bmatrix}_B X = 0$

Let $v \in V$ such that $\begin{bmatrix}v\end{bmatrix}_B = X$

$$\begin{aligned}\begin{bmatrix}T(v)\end{bmatrix}_B &= \begin{bmatrix}T\end{bmatrix}_B \begin{bmatrix}v\end{bmatrix}_B\\ &= \begin{bmatrix}T\end{bmatrix}_B X = 0\\ \therefore \quad T(v) = 0_v &\Rightarrow v = 0_v \Rightarrow X = 0\end{aligned} \tag{50}$$

To prove the reverse, assume that $\begin{bmatrix}T\end{bmatrix}_B$ is invertible. Let us prove that $Ker(T) = \{0_v\}$, which is enough to show that T is an isomorphism.

Let $v \in Ker(T), T(v) = 0_v$
$\Big[T(v)\Big]_B = 0$, ie $\Big[T\Big]_B \Big[v\Big]_B = 0$
Since $\Big[T\Big]_B$ is invertible, we must have $\Big[v\Big]_B = 0$, ie $v = 0_v$

If E is a subspace of V, $E \subseteq (E^\perp)^\perp$
Let $u \in E, \quad \forall v \in E^\perp$
$\langle u, v \rangle = 0 \Rightarrow u \in (E^\perp)^\perp, E \subseteq (E^\perp)^\perp$