MATH 323: Probability

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Contents

1 Formulas

Mean & Variance

	Mean (μ)	Variance (σ^2)
General	$\overline{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$	$\frac{1}{n-1}\sum_{i=1}^{n}(y_i - \overline{y})^2$

Expected Value & Variance

General	
Probability	$p(y)$
$E(Y)$	$\mu = \sum_y yp(y)$
$V(Y)$	$\sigma^2 = E\left[(Y-\mu)^2\right] = E(Y^2) - \mu^2$
$M_x(t)$	$E(e^{tx})$
Bernoulli	
Realization	$x = 0, 1$
$P_x(x)$	$p^x(1-p)^{1-x} = p^x q^{1-x}$
$M_x(t)$	$(1-p) + pe^t$
$E(X)$	$\sum_{x=0}^{1} xp^x q^{1-x} = p$
$V(X)$	$E(X^2) - E(X)^2 = p(1-p)$
Binomial	
Realization	$x = 0, 1, ..., n$
$P_x(x)$	$C_x^n p^x (1-p)^{n-x}$
$M_x(t)$	$(pe^t + 1 - p)^n$
$E(X)$	np
$V(X)$	$np(1-p)$
Geometric	
Realization	$x = 1, 2, ...$
$P_x(x)$	$p(1-p)^{x-1}$
$M_x(t)$	$\frac{pe^t}{1-(1-p)e^t}$
$E(X)$	$\frac{1}{p}$
$V(X)$	$\frac{1-p}{p^2}$
Negative Binomial	
Realization	$x = r, r+1, ...$
$P_x(x)$	$C_{r-1}^{x-1} p^r q^{x-r}$
$M_x(t)$	$\left[\frac{pe^t}{1-(1-p)e^t}\right]^r$
$E(X)$	$r \cdot \frac{1}{p}$
$V(X)$	$r \cdot \frac{1-p}{p^2}$

Poisson	
Realization	$x = 0, 1, 2, ...$
$P_x(x)$	$\frac{e^{-\lambda} \cdot \lambda^x}{x!}$
$M_x(t)$	$e^{\lambda(e^t - 1)}$
$E(X)$	λ
$V(X)$	λ
Exponential Distribution	
Realization	$X \sim Exp(\lambda), x > 0$
$P_x(x)$	$\lambda e^{-\lambda x}$
$M_x(t)$	$\frac{\lambda}{\lambda - t}$
$E(X)$	$\frac{1}{\lambda}$
$V(X)$	$\frac{1}{\lambda^2}$
Uniform Distribution	
Realization	$a < x < b$
$P_x(x)$	$\frac{1}{b-a} = c$
$M_x(t)$	$\frac{e^{tb} - e^{ta}}{t(b-a)}$
$E(X)$	$\frac{a+b}{2}$
$V(X)$	$\frac{(b-a)^2}{12}$
Gamma	
Realization	$x > 0$
$P_x(x)$	$\frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}$ $\frac{\Gamma(\alpha)}{\beta^\alpha} = \int_0^\infty x^{\alpha-1} e^{-\beta x} dx, \quad \Gamma(\alpha) = (\alpha - 1)!$
$M_x(t)$	$\left(\frac{\lambda}{\lambda - t}\right)^\alpha$
$E(X)$	$\alpha\beta$
$V(X)$	$\alpha\beta^2$
Beta	
Realization	$0 < x < 1$
$P_x(x)$	$\frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}$ $\alpha, \beta > 0 \quad B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$
$M_x(t)$	
$E(X)$	$\frac{\alpha}{\alpha+\beta}$
$V(X)$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

Normal	
Realization	$-\infty < x < \infty$
$P_x(x)$	$\frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/(2\sigma^2)}$
$E(X)$	μ
$V(X)$	σ^2
Standard z	$\frac{x-\mu}{\sigma}$

Chi-square is a beta distribution where $\alpha = \frac{v}{2}$ for some v, and $\beta = 2$ ($\lambda = \frac{1}{2}$). This makes $\mu = v$ and $\sigma^2 = 2v$

Combination & Permutation

- $P_r^n = \frac{n!}{(n-r)!}$
- $\binom{n}{r} = C_r^n = \frac{n!}{r!(n-r)!}$
- Permutation of n objects of k kinds, where n_i is the number of times type k occurs, is $\frac{n!}{n_1!n_2!...n_k!}$
- Circular permutations (round table) with n items taken r at a time $= \frac{P_r^n}{r}$

PMF & CDF

PMF: Probability Mass Function
Discrete r.v.: $P(x) = P(X = x)$

- $0 \leq P(x) \leq 1, \forall x$
- $\sum_x P(x) = 1$

CDF: Cumulative distribution function: $F_x(x) = P(X \leq x), x \in \mathbb{R}$

- $0 \leq F(x) \leq 1$
- $\lim_{x\to-\infty} F(x) = 0$, $\lim_{x\to\infty} F(x) = 1$
- Non-decreasing: $x_1 \leq x_2 \Rightarrow F(x_1) \leq F(x_2)$
- Right continuous

Moment Generating Function

Given $m(t) = E(e^{tY})$, $E(Y^k) = m^{(k)}(0) = \mu'_k$, $\mu'_1 = \mu$.
In finding derivatives, we can also find it against $log(m_x(t))$, as $m(0) = 1$

For simplification, note that $\sum_0^\infty \frac{f^t}{y!} = e^f$
Properties

- $M_{x+a}(t) = E[e^{(x+a)t}] = e^{at}M_x(t)$
- $M_{bx}(t) = E[e^{(bx)t}] = M_x(bt)$
- $M_{\frac{x+a}{b}}(t) = e^{\frac{a}{b}t}M_x(\frac{t}{b})$

Inequalities

- Markov: $E(x) < \infty \Rightarrow$
  $P(x \geq c) \leq \frac{E(x)}{c}, \forall c > 0$
- Tchebysheff: $E(x), Var(x) < \infty \Rightarrow$
  $P(|x - \mu| \geq k\sigma) \leq \frac{1}{k^2}$ or
  $P(|x - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}$
- Chernoff: $M_x(t)$ is a mgf of r.v. x, and $c > 0 \Rightarrow$
  $P(x \geq c) \leq e^{-tc}M_x(t)\forall t > 0$
  $P(x \leq c) \leq e^{-tc}M_x(t)\forall t < 0$

Misc

- $P(A \mid B) = \frac{A \cap B}{P(B)}$ if $P(B) > 0$
- Skewness

$$\frac{\mu^3}{\sigma^3}\begin{cases} > 0 & \text{right} \\ = 0 & \text{symmetric} \\ < 0 & \text{left} \end{cases} \tag{1}$$

- Kurtosis

$$\frac{\mu^4}{\sigma^4} - 3\begin{cases} > 0 & \text{peaked} \\ = 0 & \text{normal} \\ < 0 & \text{flat} \end{cases} \tag{2}$$

- Jensen: $f(x)$ is a convex function if:
  $a \cdot f(x_1) + (1 - a) \cdot f(x_2) \geq f(ax_1 + (1 - a)x_2)$, where $0 \leq a \leq 1$
  This also means that $E(f(x)) \geq f(E(x))$

2 Theories & Definitions

Kolmogorov Axioms

$\forall A \in \Sigma$:
m!

1. $P(A) \geq 0$ (non-negative)
2. $P(\Sigma) = 1$ (normed)

3. $P\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} P(A_j)$ when $A_i \cap A_j = \emptyset \quad \forall i \neq j$ (linearly-additive)

De-Morgan's Theorem

m!

(a) $(A \cap B)^C = A^C \cup B^C$

(b) $(A \cup B)^C = A^C \cap B^C$

Partition

Let $\{A_i\}_{i=1}^{\infty(k)}$ be a sequence of sets $A_i \leq \Omega, \forall i$. If:
m!

i. $A_i \cap A_j = \emptyset, \forall i \neq j$

ii. $\bigcup_{i=1}^{\infty(k)} A_i = \Sigma$

then we say $\{A_i\}_{i=1}^{\infty(k)}$ is a partition of sample space Ω: $\Omega = \bigcup_{j=1}^{k} A_j$

Inequalities (Boole, Bonferroni)

m!

A. Boole's Inequality
Let $A_i, A_2, ... \in \mathcal{F}$
$\Rightarrow P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i)$

B. Bonferroni's Inequality
$P\left(\bigcap_{i=1}^{\infty} A_i\right) \geq 1 - \sum_{i=1}^{\infty} P(A_i^C)$

Baye's Rule

Assume $\{A_1, A_2, ..., A_k\}$ is a partition of Ω such that $P(A_i) > 0 \forall i$
For $B \subseteq \Omega$:

C. Law of Total Probability

$$\begin{aligned} P(B) &= P\left(\bigcup_{i=1}^{k}(A_i \cap B)\right) \\ &= \sum_{i=1}^{k} P(A_i \cap B) \\ &= \sum_{i=1}^{k} P(B \mid A_i)P(A_i) \end{aligned} \tag{3}$$

D.

$$\begin{aligned} P(A_i \mid B) &= \frac{P(A_i \cap B)}{P(B)} \\ &= \frac{P(B \mid A_i)P(A_i)}{\sum_{i=1}^{k} P(B \mid A_i)P(A_i)} \end{aligned} \tag{4}$$

3 Misc

- If $\{A_i\}_{i=1}^{\infty}$ is non-decreasing, then $\lim_{n\to\infty} A_n = \bigcup_{i=1}^{\infty} A_i$
- If $\{A_i\}_{i=1}^{\infty}$ is non-increasing, then $\lim_{n\to\infty} A_n = \bigcap_{i=1}^{\infty} A_i$

Independence

A and B are independent ($A \perp\!\!\!\perp B$) if any of the following holds:

- $P(A \mid B) = P(A)$
- $P(B \mid A) = P(B)$
- $P(A \cap B) = P(A)P(B)$

Note that $\emptyset$ is independent to any event, and that
$A \perp\!\!\!\perp B \Leftrightarrow A^C \perp\!\!\!\perp B \Leftrightarrow A^C \perp\!\!\!\perp B^C \Leftrightarrow A \perp\!\!\!\perp B^C$

4 Questions

E. How many ways can 4 married couples seat themselves around a circular table if no couple can sit next to each other?

F. What is the probability of getting a full house? (3 cards of some rank and 2 cards of another rank)

G. Find the support and realization of X, where X is the number of heads, and we are tossing a fair coin three times.

5 Answers

H. $\frac{31}{105}$
Compute probabilities that i couple(s) sit next to each other for i in 1..4 and find complement of sum.

I. 0.00144
Basic combination probability:

$$\frac{(C_1^{13} \cdot C_3^4) \cdot (C_1^{12} \cdot C_2^4)}{C_5^{52}}$$

J.
- Realization = number of occurrences per sample event. eg, HHH has realization 3, and THT has realization 1
- Support = range of x. Here, it is 0 to 3
- Table of x to $P(x)$ showing the probability distribution of X:

x	$P(x)$
0	$\frac{1}{8}$
1	$\frac{3}{8}$
2	$\frac{3}{8}$
3	$\frac{1}{8}$